Coherent states, quantum gravity and the Born-Oppenheimer approximation, II: Compact Lie Groups

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In this article, the second of three, we discuss and develop the basis of a Weyl quantisation for compact Lie groups aiming at loop quantum gravity-type models. This Weyl quantisation may serve as the main mathematical tool to implement the program of space adiabatic perturbation theory in such models. As we already argued in our first article, space adiabatic perturbation theory offers an ideal framework to overcome the obstacles that hinder the direct implementation of the conventional Born-Oppenheimer approach in the canonical formulation of loop quantum gravity. Additionally, we conjecture the existence of a new form of the Segal-Bargmann-Hall "coherent state" transform for compact Lie groups G, which we prove for $G = U(1)^n$ and support by numerical evidence for $G = SU(2)$. The reason for conjoining this conjecture with the main topic of this article originates in the observation, that the coherent state transform can be used as a basic building block of a coherent state quantisation (Berezin quantisation) for compact Lie groups G. But, as Weyl and Berezin quantisation for $\mathbb{R}^{2d}$ are intimately related by heat kernel evolution, it is natural to ask, whether a similar connection exists for compact Lie groups, as well. Moreover, since the formulation of space adiabatic perturbation theory requires a (deformation) quantisation as minimal input, we analyse the question to what extent the coherent state quantisation, defined by the Segal-Bargmann-Hall transform, can serve as basis of the former.

CONTENTS

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I. INTRODUCTION

We have argued in our first article[1] that a realisation of the (time-dependent) Born-Oppenheimer approximation for multi-scale quantum dynamical systems, which are modelled by techniques used in loop quantum gravity, might be achieved along the lines of space adiabatic perturbation theory[2]. This is, because space adiabatic perturbation theory avoids some technical limitations of the original Born-Oppenheimer approach, which in turn allows us to circumvent the so-called "problem of non-commutative fast-slow coupling" (originally pointed out in the context of loop quantum gravity[1,3]). The main technical tool, necessary for a successful implementation of the ideas behind space adiabatic perturbation theory, is a Weyl quantisation associated with part of the multi-scale quantum system. More precisely, if the total quantum system is described as a coupled system decomposing into two sectors (for simplicity), one of which is called the slow sector, and the other one the fast sector (thinking of different relevant time scales), we will require the existence of a Weyl quantisation (in the sense of a real deformation quantisation) for the description of the slow subsystem. Then, if we further assume that the Weyl quantisation is scalable with a parameter ε quantifying the separation of scales between the subsystems, we can introduce a systematic perturbation theory in the sense of Born and Oppenheimer by means of dequantising the slow sector and exploiting the induced ε-dependent $\star$-product in the resulting function spaces (in analogy with standard pseudo-differential operators and their symbolic calculus).
This said, it is the primary purpose of the present article to investigate the possibility of formulating a Weyl quantisation suitable for phase spaces of the type T^*G, G a compact Lie group. The reason behind this objective is that such phase space serve as the main building block of loop quantum gravity-type models.

Before we come to the main part of the article, which is composed of three sections, let us briefly outline its structure and content:

Section II introduces a new form of the Segal-Bargmann-Hall "coherent state" transform[4] for compact Lie groups. This is motivated by the fact that unitary maps of this type provide the basis for the construction of a coherent state quantisation (also known as Berezin or Wick/Anti-Wick quantisation) of the co-tangent bundle, T^*G, of a compact Lie group G. But, before we enter into the discussion of the coherent state transform, we recall the definition of crossed product C^*-algebras, paying special attention to the transformation group C^*-algebra $C(G) \rtimes_{\mathrm{L}} G$, as the latter is intimately connected with the Hilbert space $L^2(G)$, of which the coherent states are specific elements. Moreover, in foresight of section III, introducing the C^*-algebra $C(G) \rtimes_{\mathrm{L}} G$ is convenient, because it is fundamental to the construction of the Weyl quantisation for compact Lie groups, we are aiming at. A relation between coherent state quantisation and Weyl quantisation for compact Lie groups will be established subsection III C (with regard to their application to space adiabatic perturbation theory). For completeness, we provide the essential ingredients necessary to define the coherent states for compact Lie groups introduced by Hall[4], as well.
In section III, we present possible ways to obtain Weyl (and Kohn-Nirenberg) quantisations for compact Lie groups. To this end, we follow the common philosophy[5,6] of constructing quantisations of functions (pseudo-differential operators) from left (or right) convolution kernels. Although, our

constructions, which are based on results of Turunen and Rhuzhansky[5] and Landsman[7], succeed to some extent, we are forced to deal with the dichotomy of choosing between local and global structures at various points. The latter can be traced back to the fact that the exponential map of a compact Lie group G, while still being onto, is no longer a diffeomorphims like in the case of $\mathbb{R}^{2d}$ (or nilpotent Lie groups in general). It appears, that the global formulas are easier to handle, when it comes to the composition of pseudo-differential operators (at least in the Kohn-Nirenberg formalism). But, it is the local setting, which is well adapted to deal with a semi-classical approximation of the commutation relations underlying the transformation group algebra $C(G) \rtimes_{\mathrm{L}} G$, and allows for a more direct analogy with the original treatment of space adiabatic perturbation theory in[2]. The main difficulty with the global formulas can be reduced to a lack of scale transformations compatible with the quantisation formulas and the algebraic structure of $C(G) \rtimes_{\mathrm{L}} G$, i.e. we are missing a simple relation between pseudo-differential operators with different values of the quantisation parameter ε (the adiabatic parameter in space adiabatic perturbation theory). Therefore, while asymptotic expansions of pseudo-differential operators are still conceivable in the global setting, a simple ordering in terms of powers of ε is bound to fail. In subsection III B and subsection III D, we further elaborate on this aspect: Namely, we relate the latter to the necessity of having a ε-scaleable Fourier transform on G. Then, we show that a partial solution can be achieved, if the so-called Stratonovich-Weyl transform for G[8] is invoked, but, that a general solution seems to be obstructed by the rigidity of the representation theory of G (integrality of highest weights). Only, in the case of $G = U(1)^n$, we are able to proceed further by, first, lifting everything to the universal covering group $\mathbb{R}^n$, and, second, passing to the Bohr compactification $\mathbb{R}^n_{\mathrm{Bohr}}$. The resulting theory of pseudo-differential operators connected with $\mathbb{R}_{\mathrm{Bohr}}$ is discussed and compared to the theory of almost-periodic pseudo-differential operators[9,10] in subsection III D, as well.
Finally, we present some concluding remarks and perspectives in section IV.

II. A NEW LOOK AT THE SEGAL-BARGMANN-HALL "COHERENT STATE" TRANSFORM

Although, this article is mainly concerned with the development of a Weyl quantisation for loop quantum gravity-type models, it was already noted in the preceding article[1] (cf. also[8,11]) that the (Stratonovich-)Weyl and Wick/Anti-Wick (de-)quantisations are closely related from a conceptual point of view (cf.[12] for a lucid overview), especially regarding their potential applicability in the context of space adiabatic perturbation theory (cf.[2,13]). In respect of their importance for Wick/Anti-Wick correspondences, we give a short review of the construction of coherent states on compact Lie Groups by Hall[4,14] and conjecture a new version of the Segal-Bargmann-Hall transform (or resolution of unity) that is free from the "dual" heat kernel measure ν on the complexification $G_{\mathbb{C}}$ Lie group G. We prove the conjecture in the case $G = U(1)$ and support it by some numerical evidence for $G = SU(2)$.
Since our constructions can be related to the so-called crossed product C^*-algebra $C(G) \rtimes_{\mathrm{L}} G$, which will also be important for the other sections, we briefly recall its construction (cf.[15,16]).

A. Locally compact groups and crossed products

Given a locally compact group G, a C^*-algebra $\mathfrak{A}$ and a (strongly) continuous representation $\alpha : G \to \mathrm{Aut}(\mathfrak{A})$, one makes the

Definition II.1:
The triple $(\mathfrak{A}, G, \alpha)$ *is called a (C^*-)*dynamical system*. A* covariant representation *of* $(\mathfrak{A}, G, \alpha)$ *is a triple* $(\mathfrak{H}, \pi, U)$ *consisting of a (non-degenerate) representation* $\pi : \mathfrak{A} \to \mathscr{B}(\mathfrak{H})$ *and (strongly) continuous unitary representation* $U : G \to U(\mathfrak{H})$*, s.t.*

$$\forall g \in G, A \in \mathfrak{A} : \ \pi(\alpha_g(A)) = U_g \pi(A) U_g^*. \tag{2.1}$$

Let us illustrate this definition by giving a few

Examples II.2:

1. Clearly, $(\mathfrak{A}, \{e\}, \mathrm{id})$ and $(\mathbb{C}, G, \mathrm{id})$ are trivial examples of dynamical systems. The covariant representations of these correspond to (non-degenerate) representations of $\mathfrak{A}$ in the former, and to strongly continuous unitary representation of G in the latter case.

2. The left action L of G on itself gives rise to a continuous representation $\alpha_{\mathrm{L}} : G \to \mathrm{Aut}(C_0(G))$ on the C^*-algebra of continuous functions on G vanishing at infinity by (cf.[15], Lemma 2.5.)

$$\forall f \in C_0(G), g \in G : \ \alpha_{\mathrm{L}}(g)(f) = L_{g^{-1}}^* f. \tag{2.2}$$

   An important covariant representation of the dynamical system $(C_0(G), G, \alpha_{\mathrm{L}})$ comes from the multiplier representation $\mathrm{M} : C_0(M) \to \mathscr{B}(L^2(G))$ and the left regular representation $\lambda : G \to U(L^2(G))$, where $L^2(G)$ is defined with respect to a (left) Haar measure. The compatibility of the pair (M, λ) reflects the covariance condition (2.1):

$$\forall f \in C_0(G), g \in G : \ \mathrm{M}(\alpha_{\mathrm{L}}(g)(f)) = \lambda_g \, \mathrm{M}(f) \lambda_g^*. \tag{2.3}$$

3. In analogy with the preceding example, we consider the C^*-algebra $C_b(\mathbb{R})$ of bounded continuous function on $\mathbb{R}$, and the (left) action τ of $\mathbb{R}$ on itself by translations. The triple $(C_b(\mathbb{R}), \mathbb{R}, (\tau^{-1})^*)$ forms a dynamical system, and the triple $(L^2(\mathbb{R}), \mathrm{M}, \lambda)$ is a covariant representation. Now, let us introduce the notation $U(x) := \mathrm{M}(e^{ix(\,\cdot\,)}),\ V(\xi) := \lambda_{-\xi},\ x, \xi \in \mathbb{R}$. This family of unitary operators in $L^2(\mathbb{R})$ satisfies the canonical commutation relations in Weyl form:

$$\begin{aligned} V(\xi)U(x)V(\xi)^* &= \lambda_{-\xi} \, \mathrm{M}(e^{ix(\,\cdot\,)})\lambda_{\xi} \\ &= \mathrm{M}(\alpha_{\mathrm{L}}(-\xi)(e^{ix(\,\cdot\,)})) \\ &= \mathrm{M}(e^{ix((\,\cdot\,)+\xi)}) \\ &= e^{ix\xi} \, \mathrm{M}(e^{ix(\,\cdot\,)}) \\ &= e^{ix\xi} U(x). \end{aligned} \tag{2.4}$$

   Thus, we obtain the Weyl algebra of 1-particle quantum systems as a dynamical system.

The importance of example 2 will become clear after the definition of the crossed product C^*-algebra $\mathfrak{A} \rtimes_\alpha G$ associated with a dynamical system, and the statement of theorem II.7, which will be related to our conjecture of a new Segal-Bargmann-Hall transform for compact Lie groups. But, before we turn to the definition of the crossed product, we add an remark on dynamical systems with commutative $\mathfrak{A}$ (cf.[15], especially Proposition 2.7.).

Remark II.3:

The Gelfand-Naimark theorem (cf.[16]) tells us that $\mathfrak{A} \cong C_0(X)$, where X is the set of characters of $\mathfrak{A}$ with the locally compact Hausdorff weak*-topology. Thus, the dynamical system $(\mathfrak{A}, G, \alpha)$ is isomorphic to the dynamical system $(C_0(X), G, \alpha)$. But, a dynamical system of the form $(C_0(X), G, \alpha)$ comes from a left G-space (X, G), which is why these are called *transformation group C^*-algebras*, and we may consider the fibration, $p : X \to G\backslash X$, of X over the space of (left) orbits $G\backslash X$. Since every G-orbit $G \cdot x, x \in X$ can be identified with a quotient G/H_x, where H_x is the stabiliser subgroup of $x \in X$, it is possible to associate dynamical systems $(C_0(G/H_{G\cdot x}), G, \alpha_{G/H_{G\cdot x}}),\ G \cdot x \in G \setminus X$ with sufficiently regular fibrations.
Interestingly, example 3, above, tells us that the associated $\mathbb{R}$-space is not $\mathbb{R}$ acting on itself by translations, but rather $(\beta\,\mathbb{R}, \mathbb{R})$, i.e. an action of $\mathbb{R}$ on its Stone-Čech compactification $\beta\,\mathbb{R}$, since $C_b(\mathbb{R}) \cong C_0(\beta\,\mathbb{R})$.

Coming to the discussion of the crossed product of a dynamical system $(\mathfrak{A}, G, \alpha)$, we indicate that it will be a C^*-algebra, denoted by $\mathfrak{A} \rtimes_\alpha G$, such that its (non-degenerate) representations correspond in a one-to-one fashion to covariant representations of the dynamical system. Furthermore, the crossed product is built in close analogy with the *group C^*-algebra $C^*(G)$*, which turns out to be the special case $\mathbb{C} \rtimes_{\mathrm{id}} G$.

Definition II.4 (cf.[16], Definition 2.7.2.):

*Given a dynamical system $(\mathfrak{A}, G, \alpha)$, we denote by dg and Δ, respectively, a (left) Haar measure and its modular function on G. The completion of the pre-Banach *-algebra $C_c(G, \mathfrak{A})$, equipped with*

1. *(multiplication, twisted convolution)*

$$(x * y)(g) := \int_G x(h)\alpha_h(y(h^{-1}g))dh, \tag{2.5}$$

2. *(involution)*

$$x^*(g) := \Delta(g)^{-1}\alpha_g(x(g^{-1}))^*, \tag{2.6}$$

3. *(norm)*

$$||x||_1 := \int_G ||x(h)||_{\mathfrak{A}} dh, \quad x, y \in C_c(G, \mathfrak{A}),\ g \in G, \tag{2.7}$$

is $L^1(G, \mathfrak{A})$, the convolution Banach *-algebra *of $(\mathfrak{A}, G, \alpha)$.*

Next, we need a C^*-norm on $L^1(G, \mathfrak{A})$ to define $\mathfrak{A} \rtimes_\alpha G$.

Lemma II.5 (cf.[16], p. 138, and[15], p. 52):

$$||x|| := \sup\{||\pi(x)|| \mid \pi : L^1(G, \mathfrak{A}) \to \mathscr{B}(\mathfrak{H}_\pi) - \text{a Hilbert space representation}\} \tag{2.8}$$

defines a C^*-seminorm on $L^1(G, \mathfrak{A})$, called the universal norm*. The universal norm $||\ .\ ||$ is dominated by $||\ .\ ||_1$. The completion of $L^1(G, \mathfrak{A})$ w.r.t. $||\ .\ ||$ is a C^*-algebra.*

Definition II.6:
The C^*-algebra $\overline{L^1(G,\mathfrak{A})}^{||\,\cdot\,||}$ is called the (C^*-)crossed product*, $\mathfrak{A} \rtimes_\alpha G$, of* $(\mathfrak{A}, G, \alpha)$.

The one-to-one correspondence between covariant representations $(\mathfrak{H}, \pi, U)$ of $(\mathfrak{A}, G, \alpha)$ and (non-degenerate) representations $(\mathfrak{H}, \rho)$ of $\mathfrak{A} \rtimes_\alpha G$ is achieved via the *integrated form* of the former (cf.[15], Proposition 2.40.):

$$\rho(x) := \int_G \pi(x(h))U_h dh, \quad x \in C_c(G, \mathfrak{A}). \tag{2.9}$$

As we already mentioned above, example 2 is closely related to our conjectured new Segal-Bargmann-Hall transform for $L^2(G)$, where G is a compact Lie group, which is why we state the following theorem concerning the properties of $C_0(G) \rtimes_{\mathrm{L}} G$, and its representation coming from the covariant representation $(L^2(G), \mathrm{M}, \lambda)$.

Theorem II.7 (Stone-von Neumann, cf.[15], Theorem 4.24.):
Given a locally compact group G, we have

$$C_0(G) \rtimes_{\mathrm{L}} G \cong \mathscr{K}(L^2(G)). \tag{2.10}$$

Moreover, the integrated form of the covariant representation $(L^2(G), \mathrm{M}, \lambda)$ of $(C_0(G), G, \alpha_{\mathrm{L}})$ is a faithful irreducible representation of $C_0(G) \rtimes_{\mathrm{L}} G$ onto $\mathscr{K}(L^2(G))$[17].

B. Covariant coherent states

In this subsection, we recall the definition of the coherent states given by Hall[4,14] for compact Lie groups. To this end, let G denote an arbitrary compact Lie group ($\dim G = n$) with Lie algebra $\mathfrak{g}$. On $\mathfrak{g}$, we fix an *Ad*-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{g}}$, e.g. the negative of the Killing form, and we normalise the Haar measure dg on G to coincide with the Riemannian volume measure coming from former. The inner product on $\mathfrak{g}$ gives rise to a Laplace-Beltrami operator Δ on G, which is a Casimir operator for G, and the associated heat equation,

$$\partial_t f_t = \frac{1}{2}\Delta f_t, \tag{2.11}$$

has a fundamental solution $\rho_t,\ t > 0$, the *heat kernel*, at the identity in G. ρ_t has a series expansion in terms of representation theoretical data of G:

$$\rho_t(g) = \sum_{\pi \in \hat{G}} d_\pi e^{-\frac{t}{2}\lambda_\pi} \chi_\pi(g), \tag{2.12}$$

where $\hat{G}$ is the set of isomorphism classes of irreducible unitary representations of G, d_π is the dimension, λ_π the value of the Casimir operator Δ and χ_π the character of $\pi \in \hat{G}$. An important property of ρ_t is, that it is a strictly positive C^∞-class-function on G for $t > 0$.
The Lie group G has a unique complexification $G_{\mathbb{C}}$[18] (cf.[4,19]) with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, which is the complexification of $\mathfrak{g}$. The inner product $\langle\ ,\ \rangle_{\mathfrak{g}}$ extends to a real-valued inner product on $\mathfrak{g}_{\mathbb{C}}$ via

$$\langle X_1 + iY_1, X_2 + iY_2\rangle_{\mathfrak{g}_{\mathbb{C}}} = \langle X_1, X_2\rangle_{\mathfrak{g}} + \langle Y_1, Y_2\rangle_{\mathfrak{g}}, \quad X_1, X_2, Y_1, Y_2 \in \mathfrak{g}, \tag{2.13}$$

giving rise to left-invariant Riemannian metric on $G_{\mathbb{C}}$. As for G, the Haar measure dz on $G_{\mathbb{C}}$ is normalised w.r.t. to the Riemannian volume measure coming from $\langle\ ,\ \rangle_{\mathfrak{g}_{\mathbb{C}}}$. There is also a unique antiholomorphic antiautomorphism $\forall z \in G_{\mathbb{C}}:\ z \mapsto z^*$ extending the inversion $\forall g \in G:\ g \mapsto g^{-1}$ on G. It is related to the inversion on $G_{\mathbb{C}}$ via complex conjugation $\forall z \in G_{\mathbb{C}}:\ z^* = \bar{z}^{-1}$.
The coherent states for G are constructed as result of the observation that ρ_t admits a unique analytic continuation from G to $G_{\mathbb{C}}$ (proved in[4]). In terms of the series expansion, one makes the

Definition II.8:
The functions

$$\Psi_z^t(g) := \rho_t(g^{-1}z) = \sum_{\pi \in \hat{G}} d_\pi e^{-\frac{t}{2}\lambda_\pi} \chi_\pi(g^{-1}z), \quad g \in G, z \in G_{\mathbb{C}}. \tag{2.14}$$

are called (covariant) coherent states *for G. Here, "covariance" refers to the behaviour of Ψ_z^t under the (left) regular representation of G:*

$$\begin{aligned}(\lambda_h \Psi_z^t)(g) &= \Psi_z^t(h^{-1}g) \\ &= \rho_t((h^{-1}g)^{-1}z) \\ &= \rho_t(g^{-1}(hz)) \\ &= \Psi_{hz}^t(g).\end{aligned} \tag{2.15}$$

Clearly, this G-action on the set coherent states extends to a (simply) transitive, although non-unitary w.r.t. $L^2(G)$, $G_{\mathbb{C}}$-action.

Before we comment on further properties of these functions, we need to introduce some further notation (following closely[14]). Namely, we need an analogue of ρ_t on the $G_{\mathbb{C}}$-homogeneous space $G_{\mathbb{C}}/G$. The latter admits a unique $G_{\mathbb{C}}$-invariant Riemannian structure, which agrees at the identity coset with the restriction of $\langle\ ,\ \rangle_{\mathfrak{g}_{\mathbb{C}}}$ to $i\,\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. The analogue of ρ_t is the fundamental solution ν_t at the identity coset of the heat equation,

$$\partial_t f_t = \frac{1}{4}\Delta f_t, \tag{2.16}$$

on $G_{\mathbb{C}}/G$. ν_t can be identified with a left and right G-invariant function, also denoted ν_t, on $G_{\mathbb{C}}$, which is normalised as

$$\int_{G_{\mathbb{C}}} \nu_t(z)\ dz = \mathrm{vol}(G). \tag{2.17}$$

In addition to the structures introduced so far, we need some typical objects from the structure theory of compact Lie groups. That is, we fix a maximal torus $T \subset G$ with Lie algebra $\mathfrak{t}$, and real roots $R \subset \mathfrak{t}^* \cong \mathfrak{t}$,

$$\alpha \in R :\Leftrightarrow \alpha \neq 0, \exists 0 \neq X \in \mathfrak{g}_{\mathbb{C}} : \forall H \in \mathfrak{t}:\ [H, X] = 2\pi i \alpha(H) X. \tag{2.18}$$

Furthermore, we pick a set of positive roots R^+ and denote by $\delta = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ half the sum of the positive roots. W is the Weyl group of T, C a fundamental Weyl chamber and $\Gamma \subset \mathfrak{t}$ the kernel of the exponential map restricted to $\mathfrak{t}$.
For $G_{\mathbb{C}}$, we invoke the (right) polar decomposition $z = ge^{iX} \in G_{\mathbb{C}},\ g \in G,\ X \in \mathfrak{g}$, which gives a

diffeomorphism,

$$\Phi : T^*G \cong G \times \mathfrak{g} \longrightarrow G_{\mathbb{C}},\ \Phi(g,X) = ge^{iX}, \tag{2.19}$$

that turns the "phase space", T^*G, in a natural way into a Kähler manifold. The Haar measure dz on $G_{\mathbb{C}}$ and the Liouville measure $dg\ dX$ on $T^*G \cong G \times \mathfrak{g}$ (by right translation), the latter being the product of the Haar measure on G and the Lebesgue dX, normalised by means of $\langle\ ,\ \rangle$, on $\mathfrak{g}$, fit together in the following way (cf.[14], Lemma 5):

$$\forall f \in C_c(G_{\mathbb{C}}):\ \int_{G_{\mathbb{C}}} f(z)\ dz = \int_G \int_{\mathfrak{g}} f(ge^{iX})\ dg\ \sigma(X)\ dX. \tag{2.20}$$

Here, σ is the Ad-G-invariant function on $\mathfrak{g}$ determined by

$$\forall H \in \mathfrak{t}:\ \sigma(H) = \prod_{\alpha \in R^+} \left(\frac{\sinh \alpha(H)}{\alpha(H)}\right)^2. \tag{2.21}$$

The explicit formula for the measure $\nu_t(z)\ dz$ under (right) polar decomposition is (cf.[14], Lemma 5):

$$\nu_t(z)\ dz = \frac{1}{(\pi t)^{\frac{n}{2}}} e^{-|\delta|^2_{\mathfrak{g}^*} t} e^{-\frac{1}{t}|X|^2_{\mathfrak{g}}} \eta(X)\ dg\ dX, \tag{2.22}$$

where η is an analytic square root of σ:

$$\forall H \in \mathfrak{t}:\ \eta(H) = \prod_{\alpha \in R^+} \frac{\sinh \alpha(H)}{\alpha(H)}. \tag{2.23}$$

Finally, it is important to observe that every element $z \in G_{\mathbb{C}}$ has as decomposition of the form

$$z = ge^{iH}h, \quad g, h \in G, H \in \mathfrak{t}, \tag{2.24}$$

because every element in G is conjugate to some element in the maximal torus T.

Let us now come back to the coherent states (2.14) and their properties. Firstly, they belong to the Hilbert space $L^2(G)$, which follows from the (analytically continued) heat kernel identity (cf.[4], Theorem 6):

$$\begin{aligned} (\Psi^t_z, \Psi^t_{z'})_{L^2} &= \int_G \overline{\rho_t(g^{-1}z)} \rho_t(g^{-1}z')\ dg \\ &= \rho_{2t}(z'^{-1}\bar{z}) < \infty. \end{aligned} \tag{2.25}$$

Moreover, one has an explicit formula for the norm of the coherent states (cf.[14], Eq. 8) derived from Urakawa's Poisson summation formula for the restriction of ρ_t to the maximal torus T[20]:

$$\begin{aligned} (\Psi^t_z, \Psi^t_z)_{L^2} &= \rho_{2t}(z^{-1}\bar{z}) = \rho_{2t}((z^*z)^{-1}) \\ &\overset{(2.24)}{=} \rho_{2t}(h^{-1}e^{-2iH}h) \\ &= \rho_{2t}(e^{-2iH}), \end{aligned} \tag{2.26}$$

$$\rho_{2t}(e^{-2iH}) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\ e^{|\delta|^2_{\mathfrak{g}^*}}\ e^{\frac{1}{t}|H|^2_{\mathfrak{g}}}\ \eta(H)^{-1} \times \sum_{\gamma_0\in\Gamma\cap\overline{C}} e^{i\delta(\gamma_0)}\ e^{-\frac{1}{4t}|\gamma_0|^2_{\mathfrak{g}}}\ \frac{\sum_{\gamma\in W\cdot\gamma_0} e^{-\frac{i}{t}\langle\gamma,H\rangle_{\mathfrak{g}}}\prod_{\alpha\in R^+}\alpha\left(H+\frac{1}{2i}\gamma\right)}{\prod_{\alpha\in R^+}\alpha(H)}. \tag{2.27}$$

Secondly, the coherent states provide a resolution of unity in $L^2(G)$, thus providing a unitary transformation, the (anti-)Segal-Bargmann-Hall transform,

$$L^2(G) \to \overline{\mathscr{H}}L^2(G_{\mathbb{C}},\nu_t), \qquad \forall z\in G_{\mathbb{C}} : \Phi \mapsto (\Psi^t_z,\Phi)_{L^2}, \tag{2.28}$$

which maps square integrable functions on G isometrically onto antiholomorphic square integrable, w.r.t. $\nu_t(z)\ dz$, functions on $G_{\mathbb{C}}$:

$$\forall \Phi_1,\Phi_2 \in L^2(G):\ (\Phi_1,\Phi_2)_{L^2} = \int_{G_{\mathbb{C}}} (\Phi_1,\Psi^t_z)_{L^2}(\Psi^t_z,\Phi_2)_{L^2}\ \nu_t(z)\ dz. \tag{2.29}$$

Alternatively, we will use the mnemonic (in the weak sense, using Dirac's notation):

$$\mathbb{1} = \int_{G_{\mathbb{C}}} |\Psi^t_z\rangle\langle\Psi^t_z|\ \nu_t(z)\ dz. \tag{2.30}$$

1. A new Segal-Bargmann-Hall "coherent state" transform

Although, the coherent states (2.14) can be thought of as a generalisation of the *standard coherent states* in $L^2(\mathbb{R}^n)$ there is a certain asymmetry, already pointed out in[21], which results from the coherent states not being normalised, as would be standard in quantum physical treatments due to the need for a probabilistic interpretation of the "overlap functions" $(\Phi_1,\Phi_2)_{L^2},\ \Phi_1,\Phi_2\in L^2(G)$. Furthermore, the measure $\nu_t(z)\ dz$ is not proportional to t-scaled Liouville measure $(2\pi t)^{-n}\ dg\ dX$ as one would expect in relation to the "correspondence principle". With regard to the resolution of unity (2.30), one has in the $\mathbb{R}^n$-case ($\mathbb{R}^n_{\mathbb{C}} = \mathbb{C}^n,\ z = x+ip$):

$$\begin{aligned}\mathbb{1} &= \int_{\mathbb{C}^n} |\Psi^t_z\rangle\langle\Psi^t_z|\ \nu_t(z)\ dz \\ &= C_t\int_{\mathbb{C}^n} |\Psi^t_z\rangle\langle\Psi^t_z|\ (\langle\Psi^t_z|\Psi^t_z\rangle)^{-1}\ dz,\end{aligned} \tag{2.31}$$

where $\Psi^t_z(x) = (2\pi t)^{-\frac{n}{2}}\ e^{-\frac{1}{2t}(z-x)^2},\ \nu_t(z) = (4\pi t)^{-\frac{n}{2}}\ e^{-\frac{1}{t}\Im(z)^2},\ dz = d^nx\ 2^nd^np$ and $C_t = (4\pi t)^{-n}$. The equality between the first and second line in (2.31) follows immediately, because

$$C_t((\Psi^t_z,\Psi^t_z)_{L^2})^{-1} = \nu_t(z). \tag{2.32}$$

But, the interesting point about (2.31) is, that we already know that the first line carries over to arbitrary compact Lie groups, while the second line can be written down for arbitrary compact Lie groups as well, as it involves no additional structures. Moreover the constant C_t may be compute from the norm of the coherent states $(\Psi^t_z,\Psi^t_z)_{L^2}$ as a function of $z\in\mathbb{C}^n$, explicitly it is obtained

as an integral over the imaginary directions in $\mathbb{C}$:

$$C_t^{-1} = \int_{\mathbb{R}^n} ((\Psi_z^t, \Psi_z^t)_{L^2})^{-1}\ 2^n\ d^n \Im z. \tag{2.33}$$

Unfortunately, equation (2.32) is not valid for arbitrary compact Lie groups, but, as we will argue below, the bounds on $(\Psi_z^t, \Psi_z^t)_{L^2}$ given in[14] suffice to make sense out of an analogue of the second line of (2.31). Having made this observation, we come to the main conjecture of this section.

Conjecture II.9:
Given an arbitrary Lie group G, there exists a resolution of unity

$$\mathbb{1} = C_t \int_G \int_{\mathfrak{g}} |\Psi^t_{\Phi(g,X)}\rangle\langle\Psi^t_{\Phi(g,X)}|\ (\langle\Psi^t_{\Phi(g,X)}|\Psi^t_{\Phi(g,X)}\rangle)^{-1}\ dg\ dX, \tag{2.34}$$
$$C_t^{-1} = \mathrm{vol}(G) \int_{\mathfrak{g}} (\langle\Psi^t_{\Phi(g,X)}|\Psi^t_{\Phi(g,X)}\rangle)^{-1}\ dX \propto t^{-n}.$$

for small enough $t > 0$. For commutative G or $G = SU(2)$ (2.34) holds for all $t > 0$.

We notice, that in contrast to (2.30) the resolution of unity (2.34) lives on the phase space $T^*G \cong G \times \mathfrak{g}$, which is natural from a quantum physical perspective.

A possible strategy for a proof could be provided by the fact, that $L^2(G)$ is an irreducible representation of the transformation group C^*-algebra $C(G) \rtimes_{\mathrm{L}} G$ (see theorem II.7). Thus, if the operator defined by the right hand side of the first line of (2.34) commuted with all representatives of $C(G) \rtimes_{\mathrm{L}} G$, the conjecture would be (partly) proved by an appeal to Schur's lemma.

Before, we argue in the favour of the conjecture, and prove it for $G = U(1)$, we obtain as a corollary an extension to (connected) Lie groups of compact type, i.e. those that admit an *Ad*-invariant inner product on their Lie algebras. The structure of these Lie groups is clarified by

Proposition II.10 (cf.[22], Proposition 2.2.):
Given a connected Lie group K of compact type with Ad-invariant inner product $\langle\ ,\ \rangle$ on its Lie algebra $\mathfrak{k}$, there exists a compact connected Lie group G and natural number $n \in \mathbb{N}$, such that

$$K \cong G \times \mathbb{R}^n \tag{2.35}$$

as Lie groups and the associated Lie algebra isomorphism $\mathfrak{k} \cong \mathfrak{g} \times \mathbb{R}^n$ is orthogonal.

This implies

Corollary II.11:
The resolution of unity (2.34) holds for arbitrary (connected) Lie groups $K \cong G \times \mathbb{R}^n$ of compact type, if the coherent states for K are chosen as product states of the coherent states for G and the standard coherent states for $\mathbb{R}^n$. The Liouville measure on K is then the product measure of the Liouville measures on G and $\mathbb{R}^n$, respectively.

We conclude this section with an extended

Remark II.12 (on Conjecture II.9):
From (2.20), (2.22), (2.26) and (2.27), we have the following formula for the product of $(\Psi_z^t, \Psi_z^t)_{L^2}$

and $\nu_t(z)$:

$$(\Psi^t_z, \Psi^t_z)_{L^2}\ \nu_t(z) \overset{z=ge^{iH}h}{=} (2\pi t)^{-n}\sigma(H)^{-1} \times \sum_{\gamma_0\in\Gamma\cap\overline{C}} e^{i\delta(\gamma_0)}\ e^{-\frac{1}{4t}|\gamma_0|^2_{\mathfrak{g}}}\ \frac{\sum_{\gamma\in W\cdot\gamma_0} e^{-\frac{i}{t}\langle\gamma,H\rangle_{\mathfrak{g}}} \prod_{\alpha\in R^+}\alpha\left(H+\frac{1}{2i}\gamma\right)}{\prod_{\alpha\in R^+}\alpha(H)}. \tag{2.36}$$

Now, Hall shows in[14], Proposition 3, that the absolute value of sum over $\gamma\in W\cdot\gamma_0$ is bounded from above by an expression $P(|\gamma_0|_{\mathfrak{g}}/\sqrt{t})\prod_{\alpha\in R^+}\alpha(H)$, where P is a polynomial of degree equal to twice the number of positive roots $\#R^+$. This, immediately, leads to the conclusion that, for small enough $t>0$, we have constants $a_t,\ b_t>0$ with $\lim_{t\to 0+} a_t = 1 = \lim_{t\to 0+} b_t$ exponentially fast[23], such that

$$(2\pi t)^{-n} b_t \le (\Psi^t_{\Phi(g,X)}, \Psi^t_{\Phi(g,X)})_{L^2}\ \nu_t(X)\ \sigma(X) \le (2\pi t)^{-n} a_t. \tag{2.37}$$

This shows the equivalence of the measures $\nu_t(X)\sigma(X)\ dg\ dX$ and $((\Psi^t_{\Phi(g,X)}, \Psi^t_{\Phi(g,X)})_{L^2})^{-1}\ dg\ dX$ on $G\times\mathfrak{g}$, and due to the finiteness and positivity of a_t, b_t, we know that the integrals in (2.34) makes sense. Namely, for all $\Phi_1\in L^2(G)$ we have:

$$\begin{aligned}\frac{(2\pi t)^n}{a_t}||(\Psi^t_{(\ .\ )},\Phi_1)_{L^2}||^2_{L^2(G_{\mathbb{C}},\nu_t)} &\le ||(\Psi^t_{\Phi(\ .\ )},\Phi_1)_{L^2}||^2_{L^2(G\times\mathfrak{g},(||\Psi^t||^2_{L^2})^{-1})} \\ &\le \frac{(2\pi t)^n}{b_t}||(\Psi^t_{(\ .\ )},\Phi_1)_{L^2}||^2_{L^2(G_{\mathbb{C}},\nu_t)}\end{aligned} \tag{2.38}$$

$$\begin{aligned}\Rightarrow\qquad & (\Psi^t_{(\ .\ )},\Phi_1)_{L^2}\in\overline{\mathscr{H}}L^2(G_{\mathbb{C}},\nu_t)\\ \Leftrightarrow\qquad & (\Psi^t_{\Phi(\ .\ )},\Phi_1)_{L^2}\in\overline{\mathscr{H}}L^2(G\times\mathfrak{g},(||\Psi^t_{\Phi(\ .\ )}||^2_{L^2})^{-1})\end{aligned}$$

Next, we analyse the operator

$$A_t := C_t\int_G\int_{\mathfrak{g}} |\Psi^t_{\Phi(g,X)}\rangle\langle\Psi^t_{\Phi(g,X)}|\ (\langle\Psi^t_{\Phi(g,X)}|\Psi^t_{\Phi(g,X)}\rangle)^{-1}\ dg\ dX \tag{2.39}$$

in some detail. To this end, we introduce the representative functions

$$\forall\pi\in\hat{G},\ m,n=1,...,d_\pi:\ \langle\pi,m,n|g\rangle = \pi(g)_{mn},\quad g\in G, \tag{2.40}$$

and find, because $\langle\pi,m,n|\Psi^t_{\Phi(g,X)}\rangle = e^{-\frac{t}{2}\lambda_\pi}\pi(ge^{iX})_{mn}$,

$$\begin{aligned}&\langle\pi,m,n|A_t|\pi',m',n'\rangle\\ &\quad= C_t\int_G\int_{\mathfrak{g}}\langle\pi,m,n|\Psi^t_{\Phi(g,X)}\rangle\langle\Psi^t_{\Phi(g,X)}|\pi',m',n'\rangle\ (\langle\Psi^t_{\Phi(g,X)}|\Psi^t_{\Phi(g,X)}\rangle)^{-1}\ dg\ dX\\ &\quad= C_t\int_G\int_{\mathfrak{g}}(\rho_{2t}(e^{-i2X}))^{-1}e^{-\frac{t}{2}(\lambda_\pi+\lambda_{\pi'})}\pi(ge^{iX})_{mn}\overline{\pi'(ge^{iX})_{m'n'}}\ dg\ dX\end{aligned} \tag{2.41}$$

$$= C_t \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-\frac{t}{2}(\lambda_\pi + \lambda_{\pi'})} \sum_{k,k'=1}^{d_\pi, d_{\pi'}} \pi(e^{iX})_{kn} \overline{\pi'(e^{iX})_{k'n'}} \int_G \pi(g)_{mk} \pi'(g)_{m'k'} \; dg \; dX$$

$$= C_t \operatorname{vol}(G) \; d_\pi^{-1} \; \delta_{\pi,\pi'} \; \delta_{m,m'} \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{n'n} \; dX.$$

Defining an operator $A_\pi^t \in \operatorname{End}(V_\pi)$ by the matrix elements

$$(A_\pi^t)_{n'n} = \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{n'n} \; dX, \quad n, n' = 1, ..., d_\pi, \tag{2.42}$$

we see that it commutes with $\pi(g), g \in G$, because ρ_{2t} is an even class function and dX is Ad-invariant:

$$\begin{aligned}
(\pi(g) A_\pi^t)_{mn} &= \sum_{k=1}^{d_\pi} \pi(g)_{mk} (A_\pi^t)_{kn} = \sum_{k=1}^{d_\pi} \pi(g)_{mk} \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{kn} \; dX \qquad (2.43) \\
&= \sum_{k=1}^{d_\pi} \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-t\lambda_\pi} \pi(g e^{i2X} g^{-1})_{mk} \pi(g)_{kn} \; dX \\
&= \sum_{k=1}^{d_\pi} \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2X}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2Ad_g(X)})_{mk} \pi(g)_{kn} \; dX \\
&= \sum_{k=1}^{d_\pi} \int_{\mathfrak{g}} (\rho_{2t}(e^{-i2Ad_{g^{-1}}(X)}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{mk} \pi(g)_{kn} \; dX \\
&= \sum_{k=1}^{d_\pi} \int_{\mathfrak{g}} (\rho_{2t}(g^{-1} e^{-i2X} g))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{mk} \pi(g)_{kn} \; dX \\
&= \sum_{k=1}^{d_\pi} \int_{\mathfrak{g}} (\rho_{2t}(e^{i2X}))^{-1} e^{-t\lambda_\pi} \pi(e^{i2X})_{mk} \pi(g)_{kn} \; dX \\
&= \sum_{k=1}^{d_\pi} (A_\pi^t)_{mk} \pi(g)_{kn} = (A_\pi^t \pi(g))_{mn}.
\end{aligned}$$

Thus, by Schur's lemma we have $A_\pi^t = d_\pi^{-1} \operatorname{tr}(A_\pi^t) \mathbb{1}$, and our conjecture is equivalent to the formula:

$$\begin{aligned}
C_t^{-1} d_\pi = \operatorname{tr}(A_\pi^t) &= \int_{\mathfrak{g}} (\rho_{2t}(e^{i2X}))^{-1} e^{-t\lambda_\pi} \chi_\pi(e^{i2X}) \; dX = \int_{\mathfrak{g}} \frac{e^{-t\lambda_\pi} \chi_\pi(e^{i2X})}{\sum_{\pi' \in \hat{G}} d_{\pi'} e^{-t\lambda_{\pi'}} \chi_{\pi'}(e^{i2X})} \; dX \qquad (2.44) \\
&= \int_{\mathfrak{g}} \frac{\chi_\pi(e^{t\Delta} e^{i2X})}{\sum_{\pi' \in \hat{G}} d_{\pi'} \chi_{\pi'}(e^{t\Delta} e^{i2X})} \; dX = \int_{\mathfrak{g}} \frac{\chi_\pi(e^{t\Delta + i2X})}{\sum_{\pi' \in \hat{G}} d_{\pi'} \chi_{\pi'}(e^{\Delta + i2X})} \; dX \\
&= \int_{\mathfrak{g}} \frac{\chi_\pi(e^{\frac{(t\tau + iX)^2}{t} + \frac{X^2}{t}})}{\sum_{\pi' \in \hat{G}} d_{\pi'} \chi_{\pi'}(e^{\frac{(t\tau + iX)^2}{t} + \frac{X^2}{t}})} \; dX = \int_{\mathfrak{g}} \frac{\chi_\pi(e^{\frac{(t\tau + iX)^2}{t}})}{\sum_{\pi' \in \hat{G}} d_{\pi'} \chi_{\pi'}(e^{\frac{(t\tau + iX)^2}{t}})} \; dX
\end{aligned}$$

$$= \int_{\mathfrak{g}} \frac{\mathrm{tr}_{V_\pi}(e^{\frac{(t\tau+iX)^2}{t}})}{\mathrm{tr}_{L^2(G)}(e^{\frac{(t\tau+iX)^2}{t}})}\ dX.$$

Here, we introduced the object $\tau = \sum_{i=1}^n \tau_i \otimes \tau_i$ for some orthonormal basis $\{\tau_i\}_{i=1,\dots,n} \subset \mathfrak{g}$, such that $\tau \cdot \tau = \sum_{i,j=1}^n \tau_i \tau_j \langle \tau_i, \tau_j \rangle_{\mathfrak{g}} = \sum_{i=1}^n \tau_i^2 = \Delta$ and $\tau \cdot X = \sum_{i=1}^n \tau_i X_j \langle \tau_i, \tau_j \rangle_{\mathfrak{g}}$. From a physicist's point of view, the last line in (2.44) is especially attractive, because it resembles the average over $\mathfrak{g}$ of the Boltzmann-like distribution

$$p_{(V_\pi|L^2(G))}(X) := \frac{\mathrm{tr}_{V_\pi}(e^{\frac{(t\tau+iX)^2}{t}})}{\mathrm{tr}_{L^2(G)}(e^{\frac{(t\tau+iX)^2}{t}})},\ \sum_{\pi\in\hat{G}} d_\pi p_{(V_\pi|L^2(G))}(X) = 1 \quad X \in \mathfrak{g}\,. \tag{2.45}$$

Since characters, $\chi_\pi,\ \pi \in \hat{G}$, and the (analytically continued) heat kernel ρ_t are class functions, we may further simplify the expression (2.44) by Weyl's integration formula on $\mathfrak{g}$ (cf.[24]):

$$\begin{aligned} \forall f \in C_c(\mathfrak{g}):\ \int_{\mathfrak{g}} f(X)\ dX &= \int_{C^*} \prod_{\alpha\in R^+} \alpha(H)^2 \int_G f(Ad_g(H))\ dg\ dH \\ &= \frac{1}{|W|} \int_{\mathfrak{t}} \prod_{\alpha\in R^+} \alpha(H)^2 \int_G f(Ad_g(H))\ dg\ dH. \end{aligned} \tag{2.46}$$

Here, $C^* \subset \mathfrak{t}$ corresponds to the Weyl chamber C via $\langle\ ,\ \rangle$, $|W|$ is the number of elements in W, and dH is a suitably normalised Lebesgue measure on $\mathfrak{t}$. Thus, we find for (2.44):

$$\begin{aligned} C_t^{-1} d_\pi = \mathrm{tr}(A_\pi^t) &= \int_{\mathfrak{g}} (\rho_{2t}(e^{i2X}))^{-1} e^{-t\lambda_\pi} \chi_\pi(e^{i2X})\ dX \\ &= \frac{\mathrm{vol}(G)}{|W|} \int_{\mathfrak{t}} \left(\prod_{\alpha\in R^+} \alpha(H)^2 \right) (\rho_{2t}(e^{i2H}))^{-1} e^{-t\lambda_\pi} \chi_\pi(e^{i2H})\ dH. \end{aligned} \tag{2.47}$$

To provide further evidence for our conjecture, we will prove it for $G = U(1)$, which also implies the conjecture for $G = U(1)^n,\ n \in \mathbb{N}$. Subsequently, we give numerical results that suggest the validity of the conjecture for $G = SU(2)$.

If $G = U(1)$, we have $G_{\mathbb{C}} = \mathbb{C}^*$. The coherent state overlap function is found to be (cf.[25]):

$$\langle \Psi_\xi^t | \Psi_\xi^t \rangle = \sum_{j\in\mathbb{Z}} e^{-tj^2} e^{2jl} = \vartheta_3\left(\frac{il}{\pi} \middle| \frac{it}{\pi} \right) = \vartheta_3\left(\frac{l}{t} \middle| \frac{i\pi}{t} \right) \sqrt{\frac{\pi}{t}} e^{\frac{1}{t}l^2}, \tag{2.48}$$

where we chose coordinates $\xi = e^{i(\varphi+il)},\ (\varphi, l) \in [0, 2\pi) \times \mathbb{R}$ and ϑ_3 denotes the third Jacobi theta function. The characters of the irreducible representations of $U(1)$ can be labeled by $j \in \mathbb{Z}$, $\chi_j(\xi) = \xi^l,\ d_j = 1$, and the eigenvalues of the Casimir operator are $\lambda_j = j^2$. Putting everything together, (2.44) gives:

$$C_t^{-1} = \int_{\mathbb{R}} e^{-tj^2} e^{-2jl} \left(\vartheta_3\left(\frac{l}{t} \middle| \frac{i\pi}{t} \right) \sqrt{\frac{\pi}{t}} e^{\frac{1}{t}l^2} \right)^{-1} dl = \int_{\mathbb{R}} \sqrt{\frac{t}{\pi}} e^{-\frac{1}{t}(l+tj)^2} \left(\vartheta_3\left(\frac{l}{t} \middle| \frac{i\pi}{t} \right) \right)^{-1} dl \tag{2.49}$$

$$= \int_{\mathbb{R}} \sqrt{\frac{t}{\pi}} e^{-\frac{1}{t}l^2} \left(\vartheta_3 \left(\frac{l}{t} \middle| \frac{i\pi}{t} \right) \right)^{-1} dl = t \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} e^{(l+i\frac{\pi n}{\sqrt{t}})^2} \right)^{-1} dl.$$

The last line follows from the $\mathbb{Z}$-invariance of ϑ_3 in the first argument. Clearly, this establishes the conjecture for $G = U(1)$. A Numerical evaluation of (2.49) indicates $C_t = \frac{1}{t}$.

For $G = SU(2)$, the formulas become slightly more involved, although we end up with a 1-dimensional integral as $\mathrm{rank}(SU(2)) = 1$. The characters of irreducible representations of $SU(2)$ are labeled by $n \in \mathbb{N}$, $\chi_n(e^{2iH}) = \frac{\sinh(2np)}{\sinh(2p)}$, for some suitable coordinate p on $\mathfrak{t} \cong \mathbb{R}$. The dimension and the eigenvalue of the Casimir operator are $d_n = n$ and $\lambda_n = \frac{n^2-1}{4}$, respectively. The positive root is given by $\alpha(H) = p$. This allows for the computation of all objects involved in (2.47):

$$\begin{aligned} C_t^{-1} d_n &= \frac{2\pi^2}{2} \int_{\mathbb{R}} p^2 e^{-\frac{t}{4}(n^2-1)} \frac{\sinh(2np)}{\sinh(2p)} \left(\sum_{m\in\mathbb{N}} m e^{-\frac{t}{4}(m^2-1)} \frac{\sinh(2mp)}{\sinh 2p} \right)^{-1} dp \\ &= \frac{\pi^2}{4} \int_{\mathbb{R}} p^2 e^{-\frac{1}{t}(p-\frac{t}{2}n)^2} \left(\sum_{m\in\mathbb{Z}} m e^{-\frac{1}{t}(p-\frac{t}{2}m)^2} \right)^{-1} dp \\ &= \frac{\pi^2}{4} \int_{\mathbb{R}} \left(p + \frac{t}{2}n \right)^2 e^{-\frac{1}{t}p^2} \left(\sum_{m\in\mathbb{Z}} (m+n) e^{-\frac{1}{t}(p-\frac{t}{2}m)^2} \right)^{-1} dp \\ &= \frac{\pi^2}{4} \int_{\mathbb{R}} p^2 e^{-\frac{1}{t}(p-\frac{t}{2}n)^2} \left(e^{-\frac{1}{t}p^2} (\partial_p \vartheta_3) \left(\frac{p}{2\pi i} \middle| \frac{it}{4\pi} \right) \right)^{-1} dp \\ &= i\frac{\pi^3}{2} \int_{\mathbb{R}} p^2 e^{-\frac{1}{t}(p-\frac{t}{2}n)^2} \left(e^{-\frac{1}{t}p^2} \vartheta_3' \left(\frac{p}{2\pi i} \middle| \frac{it}{4\pi} \right) \right)^{-1} dp. \end{aligned} \tag{2.50}$$

Again, we need to prove an integral formula involving the Jacobi theta function ϑ_3. Only this time, we have do deal with the derivative of ϑ_3, which is why a simple shift $p \mapsto p + \frac{t}{2}n$ does not suffice to prove the formula. Nevertheless, a numerical evaluation of the integral (see table I and figures 1 & 2)

$$I(t,n) := 2i \int_{\mathbb{R}} p^2 e^{-\frac{1}{t}(p-\frac{t}{2}n)^2} \left(e^{-\frac{1}{t}p^2} \vartheta_3' \left(\frac{p}{2\pi i} \middle| \frac{it}{4\pi} \right) \right)^{-1} dp \tag{2.51}$$

hints at the correctness of conjecture II.9 for $G = SU(2)$, since the integral relation between $I(t,n) = nI(t,1)$ seems to be a rather strong requirement. Furthermore, table I indicates the relation $I(t,n) = t^3 I(1,n)$, and thus $I(t,n) = \frac{t^3 n}{8} \Rightarrow C_t = \frac{24}{(\pi t)^3}$.

III. ON WEYL AND KOHN-NIRENBERG CALCULI FOR COMPACT LIE GROUPS

In this section, we discuss local and global Weyl and Kohn-Nirenberg calculi for compact Lie groups G ($\dim G = n$) to provide a (pseudo-differential) framework for the Born-Oppenheimer approximation[26] or space-adiabatic perturbation theory[2,13] in loop quantum gravity (see also section

t \ n	1	2	3	4	5
1	0.125	0.25	0.375	0.5	0.625
2	1	2	3	4	5
e	~ 2.51069	~ 5.02138	~ 7.53208	~ 10.0428	~ 12.5535
π	~ 3.87578	~ 7.75157	~ 11.6274	~ 15.5031	~ 19.3789
4	8	16	24	32	40

Table I. Numerical evaluation of $I(t, n)$.

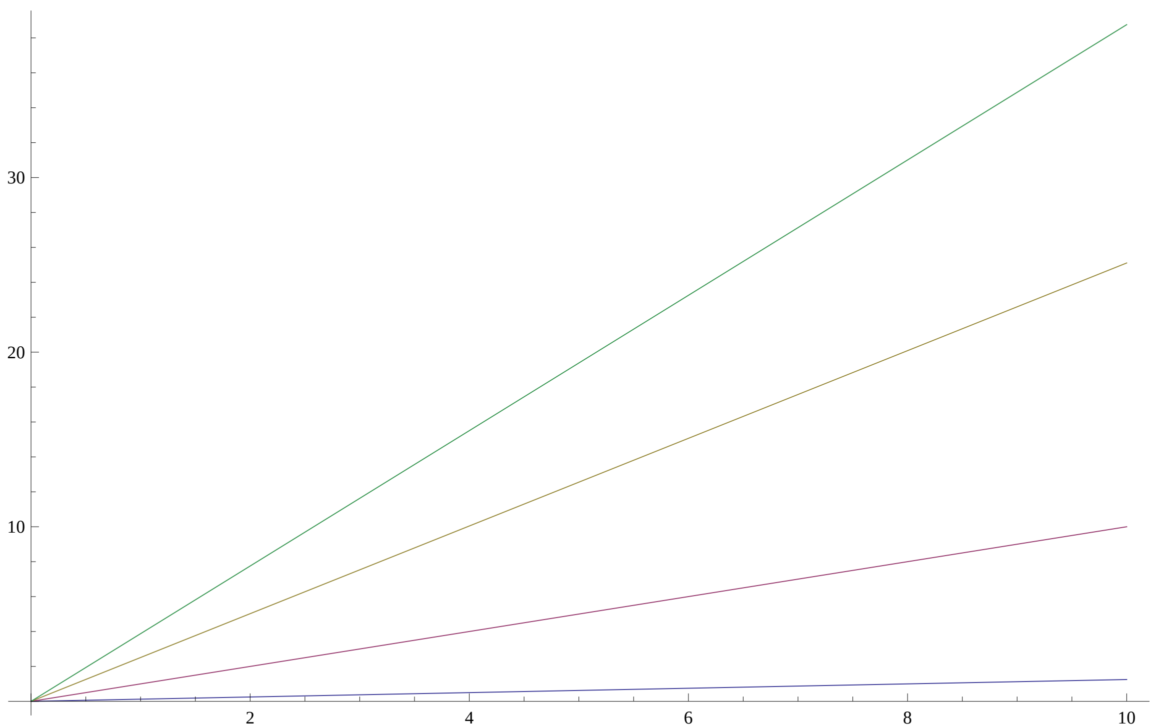

Figure 1. Plots of $I(t, n)$ (t fixed): $I(1, n)$, $I(2, n)$, $I(e, n)$ and $I(\pi, n)$ (bottom to top).

II of the third article in this series[27]). The need for local as well as global calculi is due to the fact that the exponential map is, while still onto, no longer a diffeomorphism for compact groups. Both, local and global calculi, are advantageous in certain situations: On the one hand, it is quite natural to handle the "semi-classical limit" of the (quantum) commutation relations,

$$\begin{aligned} [f, f'] &= 0, \\ [P_X, f] &= -i\varepsilon \; R_X f, \\ [P_X, P_Y] &= i\varepsilon \; P_{[X,Y]}, \end{aligned} \tag{3.1}$$

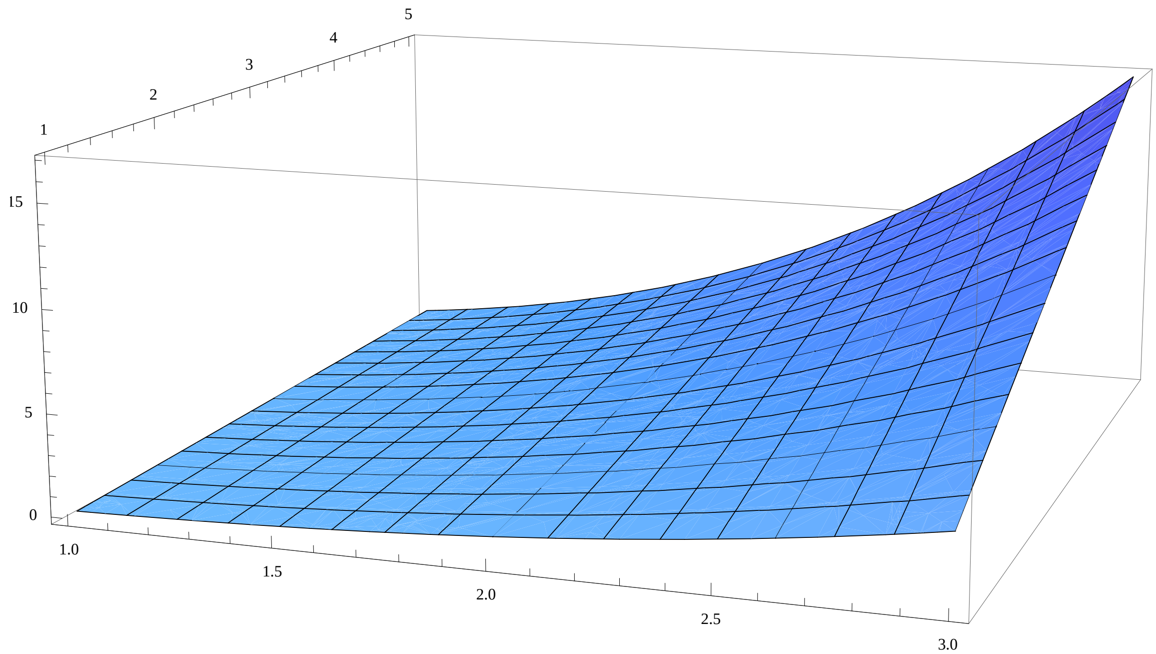

Figure 2. 3D Plot of I(n,t).

where $f, f' \in C^\infty(G)$, $X, Y \in \mathfrak{g}$ and $R_X f = \frac{d}{dt}_{|t=0} L^*_{e^{tX}} f$, in local calculi via the Baker-Campbell-Hausdorff formula, on the other hand, the composition and computation of symbols of operators is simpler for the global calculi, and the class of admissible symbols is larger.

The local calculi are based on a generalised Weyl quantisation introduced by Landsman (cf.[7,28]) based on (strict) Rieffel deformations (cf.[29,30]), while the global calculi are closely related to the Kohn-Nirenberg calculus of Ruzahnsky and Turunen (cf.[5,31]). In contrast to those existing accounts on (pseudo-differential) quantisation on compact Lie groups, we arrive at the calculi from the perspective of dequantisation of the transformation group C^*-algebra $C(G) \rtimes_{\mathrm{L}} G \cong \mathscr{K}(L^2(G))$, which is a natural quantum algebra over a single edge of loop quantum gravity (see section II of the third article in this series[27]). Furthermore, it is important to note, from the point of view of applications, that the Weyl calculi are favored over the Kohn-Nirenberg calculi, because the former are real, i.e. hermitean/self-adjoint operators tend to have hermitean/self-adjoint symbols (cf.[1]).

We start, in subsection III A 1, with the definition of the global calculi. In subsection III A 2 we introduce the local calculi and the "Paley-Wiener-Schwartz" symbol spaces $S^{K,m}_{\rho,\delta}$. For the latter, we prove a completeness result w.r.t. asymptotic expansions by adopting the method of kernel cut-off operators from the calculus of Volterra-Mellin operators (cf.[32,33]). For the global calculi we provide a reformulation in terms of the Stratonovich-Weyl transform of Figueroa, Gracia-Bondía and Várilly (cf.[8,34]) in subsection III B, which additionally gives rise to a scaled ε-scaled integral transform on G. As a byproduct, we prove strictness of the Stratonovich-Weyl quantisation.

Following this, we comment on the relation of the calculi to coherent state quantisation in subsection III C. We conclude the section with a closer look a the special case $G = U(1)$ and a possible extension

of the global calculi to $G = \mathbb{R}_{\text{Bohr}}$, where we will argue that the global calculus is suitable for dealing with symbols that are not analytic in the momenta.

A. Pseudo-Differential Operators on Compact Lie groups

Before we state the definitions for the types of pseudo-differential operators on compact Lie groups that we intend to discuss, we start with a (informal) motivation, i.e. we refrain from defining the function spaces on which the following formulae will be well-defined:
For quantum mechanics on $\mathbb{R}^n$ the commutation relations (3.1) correspond to the standard commutation relations for position and momentum,

$$\begin{aligned} [Q_i, Q_j] &= 0 = [P_i, P_j] \\ [P_i, Q_j] &= -i\varepsilon\ \delta_{i,j}, \quad \forall i,j = 1,...,n, \end{aligned} \tag{3.2}$$

which are often presented in their Weyl form obtained by considering the (formal) exponentials $W(x,\xi) := e^{i(x\cdot Q+\xi\cdot P)}$, $x,\xi \in \mathbb{R}^n$, (*Weyl elements*). In the Schrödinger representation on $L^2(\mathbb{R}^n)$ the action of the Weyl elements is defined to be:

$$(W(x,\xi)\Psi)(q) = e^{\frac{i\varepsilon}{2}x\cdot\xi}e^{i\xi\cdot q}\Psi(q+\varepsilon\xi), \quad \Psi \in L^2(\mathbb{R}^n). \tag{3.3}$$

Moreover, the Weyl elements provide a means of quantising functions σ on phase space $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ to operators A_σ on $L^2(\mathbb{R}^n)$ by Fourier transformation, i.e.:

$$A_\sigma := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} dx\ d\xi\ \mathscr{F}[\sigma](x,\xi)W(x,\xi) = \int_{\mathbb{R}^{2n}} dq\ dp\ \sigma(q,p)\hat{W}(q,p) = (\hat{W},\sigma)_{L^2(\mathbb{R}^{2n})}, \tag{3.4}$$

where we introduce the *Fourier-Weyl elements* $\hat{W}(q,p) := \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} dx\ d\xi\ e^{-i(x\cdot q+\xi\cdot p)}W(x,\xi)$, $\hat{W}(q,p)^* = \hat{W}(q,p)$, and $\mathscr{F}[\sigma](x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} dq\ dp\ \sigma(q,p)e^{-i(x\cdot q+\xi\cdot p)}$ is the Fourier transform of σ. Combining (3.3) and (3.4), we find

$$(A_\sigma\Psi)(q) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^{2n}} dx\ d\xi\ \sigma(\tfrac{1}{2}(q+x),\xi)e^{\frac{i}{\varepsilon}\xi\cdot(q-x)}\Psi(x). \tag{3.5}$$

Thus, A_σ is pseudo-differential operator on $L^2(\mathbb{R}^n)$ with *amplitude* oder *symbol* σ. In addition to the quantisation formula (3.4), the Fourier-Weyl elements also give rise to a useful formula for the dequantisation of an operator A on $L^2(\mathbb{R}^n)$, i.e. finding a symbol σ_A s.t. $A_{\sigma_A} = A$, because of the (distributional) orthogonality relations

$$\begin{aligned} \left(\frac{\varepsilon}{2\pi}\right)^n (W(x,\xi),W(x',\xi'))_{HS} &= \left(\frac{\varepsilon}{2\pi}\right)^n \operatorname{tr}(W(x,\xi)^*W(x',\xi')) = \delta^{(n)}(x-x')\delta^{(n)}(\xi-\xi') \\ (2\pi\varepsilon)^n(\hat{W}(q,p),\hat{W}(q',p'))_{HS} &= (2\pi\varepsilon)^n \operatorname{tr}(\hat{W}(q,p)^*\hat{W}(q',p')) = \delta^{(n)}(q-q')\delta^{(n)}(p-p'). \end{aligned} \tag{3.6}$$

Applying (3.6) to the product $A_\rho = A_\sigma A_\tau$ of two operators A_σ and A_τ, we find the well-known formula for the *twisted or Moyal product* $\rho = \sigma \star_\varepsilon \tau$ of the symbols σ and τ, and its asymptotic

expansion (cf.[11]):

$$\rho(q,p) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} dq'\, dp'\, e^{i(q\cdot q' + p\cdot p')} \int_{\mathbb{R}^{2n}} dq''\, dp''\, \mathscr{F}[\sigma](q'',p'')\mathscr{F}[\tau](q'-q'',p'-p'') e^{\frac{i\varepsilon}{2}(q'\cdot p''-p'\cdot q'')}$$
$$\sim \exp\left(-\frac{i\varepsilon}{2}(\nabla_p\cdot\nabla_x - \nabla_q\cdot\nabla_\xi)\right)_{|(q,p)=(x,\xi)} \sigma(q,p)\tau(x,\xi). \tag{3.7}$$

Clearly, to arrive at (3.7) we need to evaluate the *tri-kernel* $\mathrm{tr}(\hat{W}(q,p)^*\hat{W}(q',p')\hat{W}(q'',p''))$, which is possible, because linear combinations of (Fourier-)Weyl elements are closed under products.

Thus, we conclude that the structures required for a theory of pseudo-differential operators adapted to quantum systems defined by the commutation relations (3.1) or their exponential form (which we recognize as covariant representation of $(C(G), G, \alpha_{\mathrm{L}})$[35,36]),

$$\begin{aligned} [f,f'] &= 0, \\ U_g f U_g^* &= L_{g^{-1}}^* f \\ U_g U_{g'} &= U_{gg'}, \qquad f,f'\in C(G),\ g,g'\in G, \end{aligned} \tag{3.8}$$

are a set of (Fourier-)Weyl elements, closed under product if possible, and a notion of Fourier transform. Moreover, to obtain a practical calculus for the application of the Born-Oppenheimer scheme and space-adiabatic perturbation theory we should require the existence of a formula similar to (3.7) for symbols of operator products, which admits a suitable asymptotic expansion.

Let us also add a short comment on the choice of Weyl elements (3.3): Namely, we could make the alternative definitions

$$W_R(x,\xi) := e^{ix\cdot Q}e^{i\xi\cdot P}, \qquad\qquad W_L(x,\xi) := e^{i\xi\cdot P}e^{ix\cdot Q}, \tag{3.9}$$

which lead to the Kohn-Nirenberg pseudo-differential operators A^L_σ, A^R_σ associated with a symbol σ (cf.[11]), which are standard in treatments of pseudo-differential operators on manifolds (cf.[5,37]):

$$(A^R_\sigma\Psi)(q) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^{2n}} dx\ d\xi\ \sigma(q,\xi)e^{\frac{i}{\varepsilon}\xi\cdot(q-x)}\Psi(x), \tag{3.10}$$

$$(A^L_\sigma\Psi)(q) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^{2n}} dx\ d\xi\ \sigma(x,\xi)e^{\frac{i}{\varepsilon}\xi\cdot(q-x)}\Psi(x). \tag{3.11}$$

Although, the Kohn-Nirenberg pseudo-differential operators generalise in a much more straightforward manner to manifolds, because (3.10) and (3.11) can be localised in the position variable q[12], and thus to general (compact) Lie groups, they are disadvantageous w.r.t. to the Born-Oppenheimer scheme and space-adiabatic perturbation theory, because they do not provide a real quantisation in contrast to (3.5), i.e. $\left(A^{R/L}_\sigma\right)^* \neq A^{R/L}_{\sigma^*}$, due to the asymmetric treatment of Q and P.

1. A global calculus

Having identified the ingredients necessary for the definition of pseudo-differential operators, we will now explain how these are realized for a compact Lie group G in a global fashion. With

regard to the notation, we stick to section II. As noted above, the commutation relations (3.1) correspond in exponential form (3.8) to a covariant representation of $(C(G), G, \alpha_{\mathrm{L}})$, and thus to the transformation group C^*-algebra $C(G) \rtimes_{\mathrm{L}} G$. In the (faithful) integrated representation ρ_{L} on $L^2(G)$ a function $F \in C(G \times G) \subset C(G) \rtimes_{\mathrm{L}} G$ acts according to

$$\begin{aligned}(\rho_{\mathrm{L}}(F)\Psi)(g) &= \int_G dh\ F(h,g)(U_h\Psi)(g) = \int_G dh\ F(h,g)\Psi(h^{-1}g) \\ &= \int_G dh\ F(gh^{-1},g)\Psi(h), \quad \Psi \in L^2(G).\end{aligned} \tag{3.12}$$

Clearly, (3.12) still to makes sense as a continuous operator $\rho_{\mathrm{L}}(F) : C^\infty(G) \to \mathscr{D}'(G)$ for $F \in \mathscr{D}'(G \times G) \cong \mathscr{D}'(G) \hat{\otimes} \mathscr{D}'(G)$, and if we restrict F to be in $\mathscr{D}'(G, C^\infty(G)) \cong \mathscr{D}'(G) \hat{\otimes} C^\infty(G)$ it defines a continuous operator on $C^\infty(G)$[38]. Therefore, it is possible to make

Definition III.1:
For $F^R(\pi,m,n;h) := \pi(\ .\ )_{mn}\delta_h,\ F^L(\pi,m,n;h) := \pi(h^{-1}\ .\ )_{mn}\delta_h \in \mathscr{D}'(G, C^\infty(G))$, where $\pi \in \hat{G}$, $m,n = 1,...,d_\pi$ is a unitary irreducible representation, we obtain the operators

$$\begin{aligned}&\left(\rho_{\mathrm{L}}\left(F^R(\pi,m,n;h)\right)\Psi\right)(g) = \pi(g)_{mn}\Psi(h^{-1}g), \\ &\left(\rho_{\mathrm{L}}\left(F^L(\pi,m,n;h)\right)\Psi\right)(g) = \pi(h^{-1}g)_{mn}\Psi(h^{-1}g), \quad \Psi \in C^\infty(G),\end{aligned} \tag{3.13}$$

which extend to operators on $L^2(G)$ by continuity.

In view of (3.9), these operators generalise the Weyl elements used in the definition of the Kohn-Nirenberg pseudo-differential operators (3.10) and (3.11). In the following, we will restrict attention to the (right) elements $F^R(\pi,m,n;h)$, although all statements hold in slightly modified form for $F^L(\pi,m,n;h)$, as well. The analogy with (3.10) and (3.11) becomes even clearer, when we consider the Weyl elements in context of the (global) Fourier transform between G and its (unitary) dual $\hat{G}$[39], and its inverse,

$$\begin{aligned}\mathscr{F}[\Psi](\pi) = \hat{\Psi}(\pi) &:= \int_G dg\ \Psi(g)\pi(g), \quad \Psi \in L^2(G), \\ \mathscr{F}^{-1}[\Phi](g) = \check{\Phi}(g) &:= \sum_{\pi \in \hat{G}} d_\pi \operatorname{tr}(\pi(g)^*\Phi(\pi)), \quad \Phi \in L^2(\hat{G}) :\end{aligned} \tag{3.14}$$

We define the Fourier-Weyl elements

$$\begin{aligned}\hat{F}^R(g;\pi,m,n) &:= \sum_{\pi' \in \hat{G}} d_{\pi'} \sum_{m',n'=1}^{d_{\pi'}} \overline{\pi'(g)_{m'n'}} \int_G dh\ \overline{\pi(h)_{mn}} F^R(\pi',m',n';h) \\ &= \delta_g \overline{\pi(\ .\ )_{mn}} \in C^\infty(G, \mathscr{D}'(G)).\end{aligned} \tag{3.15}$$

Then, for $\sigma \in \hat{\mathscr{D}}'(\hat{G}, C^\infty(G)) = \mathscr{F}_1(\mathscr{D}'(G, C^\infty(G)))$, we have

$$F^R_\sigma(h,g) = \sum_{\pi \in \hat{G}} d_\pi \sum_{m,n=1}^{d_\pi} \int_G dg'\ \sigma(\pi,g')_{mn}\hat{F}^R(g';\pi,m,n)(h,g) \tag{3.16}$$

$$= \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\pi(h)^*\sigma(\pi,g)) = \check{\sigma}^1(h,g),$$

which is analogous to (3.4), and continuous to be well-defined in the distributional sense for $\sigma \in \mathscr{F}_1(\mathscr{D}'(G\times G))$. The representation of $\check{\sigma}$ on $C^\infty(G)$ via ρ_{L} leads to the pseudo-differential operators of Ruzhansky and Turunen[5] (up to the fact that those authors employ the right convolution algebra $C(G) \rtimes_{\mathrm{R}} G$):

$$\begin{aligned}\left(\rho_{\mathrm{L}}\left(F^R_\sigma\right)\Psi\right)(g) &= \int_G dh\ F^R_\sigma(gh^{-1},g)\Psi(h) \\ &= \int_G \check{\sigma}(gh^{-1},g)\Psi(h) = \sum_{\pi\in\hat{G}} d_\pi \operatorname{tr}(\pi(g)^*\sigma(\pi,g)\hat{\Psi}(\pi)).\end{aligned} \tag{3.17}$$

As a consequence of the Peter-Weyl theorem and the fact that (left) convolution is well-defined on $\mathscr{D}'(G)$ the (Fourier-)Weyl elements satisfy (distributional) orthogonality relations:

$$\begin{aligned}\operatorname{tr}\left(\rho_{\mathrm{L}}\left(F^R(\pi,m,n;h)^*\right)\rho_{\mathrm{L}}\left(F^R(\pi',m',n';h')\right)\right) &= d_\pi^{-1}\delta_{\pi,\pi'}\delta_{m,m'}\delta_{n,n'}\delta_h(h'), \\ \operatorname{tr}\left(\rho_{\mathrm{L}}\left(F^R(g;\pi,m,n)^*\right)\rho_{\mathrm{L}}\left(F^R(g';\pi',m',n')\right)\right) &= \delta_{g'}(g)d_\pi^{-1}\delta_{\pi,\pi'}\delta_{m,m'}\delta_{n,n'},\end{aligned} \tag{3.18}$$

where the adjoint $(\ .\ )^*$ has to be taken the sense of (2.6). Furthermore, the linear span of Weyl elements is closed under products due to complete reducibility of unitary representations of G. in fact, we have:

$$\begin{aligned}&\left(F^R(\pi,m,n;h) *_{\mathrm{L}} F^R(\pi',m',n';h')\right)(g,g') \\ &= \pi(g')_{mn}\pi'(h^{-1}g')\delta_{hh'}(g) = \sum_{k'=1}^{d_{\pi'}} \pi'(h^{-1})_{m'k'}\pi(g')_{mn}\pi'(g')_{k'n'}\delta_{hh'}(g) \\ &= \sum_{k'=1}^{d_{\pi'}} \pi'(h^{-1})_{m'k'} \sum_{\pi''\in\hat{G}} \sum_{s=0}^{N^{\pi''}_{\pi,\pi'}} \sum_{M,N=1}^{d_{\pi''}} C(\pi,m;\pi',k'|\pi'',M;s)\overline{C(\pi,n;\pi',n'|\pi'',N;s)}\pi''(g')_{MN}\delta_{hh'}(g) \\ &= \sum_{k'=1}^{d_{\pi'}} \pi'(h^{-1})_{m'k'} \sum_{\pi''\in\hat{G}} \sum_{s=1}^{N^{\pi''}_{\pi,\pi'}} \sum_{M,N=1}^{d_{\pi''}} C(\pi,m;\pi',k'|\pi'',M;s)\overline{C(\pi,n;\pi',n'|\pi'',N;s)} \\ &\qquad\qquad \times F^R(\pi'',M,N;hh')(g,g'),\end{aligned} \tag{3.19}$$

where we denote by the C's the Clebsch-Gordan coefficients associated with the decomposition

$$\pi\otimes\pi' \cong \bigoplus_{\pi''\in\hat{G}} \bigoplus_{s=1}^{N^{\pi''}_{\pi,\pi'}} \pi'' \ \text{ with multiplicities } N^{\pi''}_{\pi,\pi'} \in \mathbb{N}_0. \tag{3.20}$$

Dequantisation of a continuous operator A on $C^\infty(G)$ takes the form:

$$\sigma_A(\pi,g)_{mn} := \operatorname{tr}\left(\rho_{\mathrm{L}}\left(\hat{F}^R(g;\pi,m,n)^*\right)A\right), \tag{3.21}$$

which agrees with the formula of Ruzhansky and Turunen[5] for $A = \rho_{\mathrm{L}}(F),\ F \in \mathscr{D}'(G, C^\infty(G))$:

$$\sigma_F(\pi, g)_{mn} = \mathrm{tr}\left(\rho_{\mathrm{L}}\left(\hat{F}^R(g;\pi,m,n)^*\right)\rho_{\mathrm{L}}(F)\right) = \int_G dh\ F(h,g)\pi(h)_{mn} = \hat{F}(\pi,g)_{mn}. \tag{3.22}$$

By the Schwartz kernel theorem (cf.[5,40]) this covers already the general case of continuous operators on $C^\infty(G)$.

Before we try to deform the Kohn-Nirenberg quantisation, defined so far, to obtain a Weyl quantisation, i.e. a real quantisation, we collect some of the properties of the former in a

Proposition III.2:

Given continuous operators $A, B : C^\infty(G) \to \mathscr{D}'(G)$ with $F_A, F_B \in \mathscr{D}'(G \times G)$, the symbols $\sigma_A, \sigma_B \in \mathscr{F}_1(\mathscr{D}'(G \times G))$ satisfy:

1. *The map $A \mapsto \sigma_A$ is a linear homeomorphism between $L(C^\infty(G), \mathscr{D}'(G))$ and $\hat{\mathscr{D}}'(\hat{G} \times G)$ with $\sigma_{\mathbb{1}}(\pi, g) = \pi(e)$.*
2. *$\sigma_{A^*}(\pi, g) = \sigma_{F_A^*}(\pi, g) = \sigma_{\alpha_{\mathrm{L}}^{-1}(F_A)}(\pi, g)^*$, where $(\alpha_{\mathrm{L}}^{-1}(F))(h, g) = F(h, hg),\ F \in \mathscr{D}'(G \times G)$.*
3. *$\sigma_{U_h A U_h^*}(\pi, g) = \sigma_{(\alpha_{h^{-1}} \times L_{h^{-1}})^* F_A}(\pi, g) = \pi(h)\sigma_A(\pi, h^{-1}g)\pi(h)^*$, where $\alpha_h(g) = hgh^{-1}$.*
4. *If $U^R : G \to U(L^2(G))$ denotes the right regular representation, i.e. $(U_h^R \Psi)(g) = \Psi(gh)$ for $\Psi \in L^2(G),\ h, g \in G$, we have: $\sigma_{U_h^R A U_h^{R*}}(\pi, g) = \sigma_{(\mathrm{id} \times R_h)^* F_A}(\pi, g) = \sigma_A(\pi, gh)$.*
5. *$(A, B)_{HS} = \mathrm{tr}(A^* B) = (F_A, F_B)_{L^2(G \times G)} = (\sigma_A, \sigma_B)_{L^2(\hat{G} \times G)}$, if A, B are Hilbert-Schmidt operators. Moreover, the maps $HS(L^2(G)) \ni A \mapsto F_A \in L^2(G \times G)$ and $HS(L^2(G)) \ni A \mapsto \sigma_A \in L^2(\hat{G} \times G)$ are unitary.*
6. *$\sigma_{AB}(\pi, g) = \int_G dh\ F_A(h, g)\pi(h)\sigma_B(\pi, h^{-1}g)$, if $A, B : C^\infty(G) \to C^\infty(G)$.*
7. *The symbol σ_A can be computed by*

$$\sigma_A(\pi, g) = (\pi A \pi^*)(g) = \pi(g) \int_G dh\ F_A(gh^{-1}, g)\pi(h)^*. \tag{3.23}$$

Proof:

1. The linearity of $A \mapsto \sigma_A$ and $\sigma_{\mathbb{1}}(\pi, g) = \pi(e)$ are evident from the definition. The homeomorphism property follows from the Schwartz kernel theorem and the fact that the (partial) Fourier transform sets up a homeomorphism between $\mathscr{D}'(G \times G)$ and $\hat{\mathscr{D}}'(\hat{G} \times G)$.
2. For $\Psi \in C^\infty(G)$, we have

$$\begin{aligned}(\sigma_{A^*}(\pi), \Psi) &= \int_G dg\ \int_G dh\ F_{A^*}(h, g)\pi(h)\Psi(g) = \int_G dg\ \int_G dh\ F_A^*(h, g)\pi(h)\Psi(g) \\ &= \int_G dg\ \int_G dh\ \overline{F_A(h^{-1}, h^{-1}g)}\pi(h)\Psi(g) = \int_G dg\ \int_G dh\ \overline{F_A(h, hg)}\pi(h)^*\Psi(g) \\ &= \int_G dg\ \left(\int_G dh\ F_A(h, hg)\pi(h)\right)^* \Psi(g)\end{aligned} \tag{3.24}$$

$$= \int_G dg \ \left(\int_G dh \ \alpha_{\mathrm{L}}^{-1}(F_A)(h,g)\pi(h) \right)^* \Psi(g).$$

3. Referring to the definition of $A = \rho_{\mathrm{L}}(F_A)$, we find:

$$U_h A U_h^* = U_h \rho_{\mathrm{L}}(F_A) U_h^* = \rho_{\mathrm{L}}((\alpha_{h^{-1}} \times L_{h^{-1}})^* F_A). \tag{3.25}$$

   Therefore, we have:

$$\begin{aligned} \sigma_{U_h A U_h^*}(\pi, g) &= \sigma_{(\alpha_{h^{-1}} \times L_{h^{-1}})^* F_A}(\pi, g) = \int_G dh' \ (\alpha_{h^{-1}} \times L_{h^{-1}})^* F_A(h', g)\pi(h') \\ &= \int_G dh' F_A(h', h^{-1}g)\pi(\alpha_h(h')) = \pi(h) \int_G dh' \ F_A(h', h^{-1}g)\pi(h')\pi(h)^* \\ &= \pi(h)\sigma_A(\pi, h^{-1}g)\pi(h)^*. \end{aligned} \tag{3.26}$$

4. This follows along the same lines as 3.

5. It is well known (cf.[41]), that A, B are Hilbert-Schmidt operators if and only if $F_A, F_B \in L^2(G \times G)$, which gives the second equality and the unitarity of $A \mapsto F_A$. The third equality and the unitarity of $A \mapsto \sigma_A$ follow, because the (partial) Fourier transform is a unitary map from $L^2(G \times G)$ to $L^2(\hat{G} \times G)$.

6. By assumption the product AB is well-defined and gives rise to a (left) convolution kernel F_{AB}, which is found from

$$AB = \rho_{\mathrm{L}}(F_A)\rho_{\mathrm{L}}(F_B) = \rho_{\mathrm{L}}(F_A *_{\mathrm{L}} F_B). \tag{3.27}$$

   The formula for σ_{AB} follows from direct computation:

$$\begin{aligned} \sigma_{AB}(\pi, g) &= \int_G dh \ (F_A *_{\mathrm{L}} F_B)(h, g)\pi(h) \\ &= \int_G dh \ \int_G dh' F_A(h', g) F_B(h'^{-1}h, h'^{-1}g)\pi(h) \\ &= \int_G dh \ \int_G dh' F_A(h', g)\pi(h') F_B(h'^{-1}h, h'^{-1}g)\pi(h'^{-1}h) \\ &= \int_G dh' \ F_A(h', g)\pi(h') \int_G dh \ F_B(h, h'^{-1}g)\pi(h) \\ &= \int_G dh \ F_A(h, g)\pi(h)\sigma_B(\pi, h^{-1}g). \end{aligned} \tag{3.28}$$

7. Employing (3.22), we find:

$$\begin{aligned} \hat{F}_A(\pi, g) &= \int_G dh \ F_A(h, g)\pi(h) = \pi(g) \int_G dh \ F_A(h, g)\pi(g^{-1}h) \\ &= \pi(g) \int_G dh \ F_A(gh, g)\pi(h) \end{aligned} \tag{3.29}$$

$$= \pi(g) \int_G dh \; F_A(gh^{-1}, h)\pi(h)^* = (\pi A \pi^*)(g). \qquad \square$$

The last property is extremely useful in the actual computation of symbols σ_A of operators $A : C^\infty(G) \to \mathscr{D}'(G)$, as it does not require the computation of the (left) convolution kernel F_A. To see how this works, we compute the symbols of $P_X = -i\varepsilon R_X,\ X \in \mathfrak{g}$, and $f \in C^\infty(G)$:

$$\begin{aligned} \sigma_{P_X}(\pi, g) &= (\pi P_X \pi^*) = \pi(g)(P_X\pi^*)(g) = -i\varepsilon\pi(g)\frac{d}{dt}_{|t=0}\pi(e^{tX}g)^* \\ &= -i\varepsilon\pi(g)\frac{d}{dt}_{|t=0}\pi(g)^*\pi(e^{-tX}) = i\varepsilon d\pi(X), \\ \sigma_f(\pi, g) &= (\pi f \pi^*)(g) = \pi(g)f(g)\pi(g)^* = f(g)\mathbb{1}_{V_\pi}. \end{aligned} \tag{3.30}$$

Combining this with the sixth property, gives rise to the commutation relations (3.1):

$$\begin{aligned} \sigma_{ff'}(\pi, g) &= \pi(g)(ff'\pi^*)(g) = f(g)f'(g)\mathbb{1}_{V_\pi} = \sigma_f(\pi, g)\sigma_{f'}(\pi, g), \\ \sigma_{P_X f} &= \pi(g)(P_X f\pi^*)(g) = \pi(g)(P_X\pi^*)(g)f(g) - i\varepsilon(R_X f)(g)\mathbb{1}_{V_\pi} \\ &= \sigma_{P_X}(\pi, g)\sigma_f(\pi, g) - i\varepsilon\sigma_{R_X f}(\pi, g), \\ \sigma_{P_X P_Y}(\pi, g) &= \pi(g)(P_X P_Y \pi^*)(g) = i\varepsilon\pi(g)(P_X\pi^*)(g)d\pi(Y) = -\varepsilon^2 d\pi(X)d\pi(Y) = \sigma_{P_X}\sigma_{P_Y}. \end{aligned} \tag{3.31}$$

The second property of the above proposition gives a measure to which extent the quantisation $\hat{\mathscr{D}}'(\hat{G} \times G) \ni \sigma \mapsto \rho_{\mathrm{L}}(F^R_\sigma) \in L(C^\infty(G), \mathscr{D}'(G))$ fails to be real. Interestingly, there is a simple way to cure this, if we were allowed to take square roots in G, which is indeed possible for compact Lie groups by means of the exponential map $\exp : \mathfrak{g} \to G$, as the latter is onto (cf.[19]). Moreover, we can define $\sqrt{g},\ \forall g \in G$, s.t. $\sqrt{g^{-1}} = \sqrt{g}^{-1}$, but there is also a price to pay: Namely, $\sqrt{.} : G \to G$ is in general not a smooth homomorphism, but only a measurable map.

Definition III.3:
For $\sigma \in \hat{\mathscr{D}}'(\hat{G} \times G)$, s.t. $\alpha_{\mathrm{L}}^{\frac{1}{2}}\left(F^R_\sigma\right)(h, g) = F^R_\sigma(h, \sqrt{h}^{-1}g)$ is in $\mathscr{D}'(G \times G)$, e.g. if $\sqrt{.}$ is smooth on $\operatorname{sing\,supp}_1(F^R_\sigma)$, *we define the Weyl quantisation F^W_σ of σ by*

$$F^W_\sigma(h, g) := \alpha_{\mathrm{L}}^{\frac{1}{2}}\left(F^R_\sigma\right)(h, g). \tag{3.32}$$

The Weyl elements $F^W(\pi, m, n; h)$ are the Weyl quantisations of the symbols

$$(\sigma_{(\pi,m,n;h)}(\pi', h'))_{m'n'} := d_\pi^{-1}\delta_{\pi,\pi'}\delta_{m',m}\delta_{n,n'}\delta_h(h'), \tag{3.33}$$

where $\pi, \pi' \in \hat{G},\ m, n = 1, ..., d_\pi,\ m', n' = 1, ..., d_{\pi'},\ h, h' \in G$. Explicitly, we have

$$\begin{aligned} F^W(\pi, m, n; h)(h', g) &= \pi(\sqrt{h}^{-1}g)_{mn}\delta_h(h') \\ &= \sum_{k=1}^{d_\pi}\pi(\sqrt{h}^{-1})_{mk}\pi(g)_{kn}\delta_h(h') \\ &= \sum_{k=1}^{d_\pi}\pi(\sqrt{h}^{-1})_{mk}F^R(\pi, k, n; h)(h', g). \end{aligned} \tag{3.34}$$

The following lemma shows that the Weyl quantisation is real, and that the Weyl element satisfies appropriate orthogonality relations.

Lemma III.4:
For $\sigma \in \hat{\mathscr{D}}'(\hat{G} \times G)$ as in definition III.3, we have

$$\rho_{\mathrm{L}}\left(F^W_\sigma\right)^* = \rho_{\mathrm{L}}\left(F^{W*}_\sigma\right) = \rho_{\mathrm{L}}\left(F^W_{\sigma^*}\right). \tag{3.35}$$

Moreover, the Weyl elements satisfy

$$\mathrm{tr}\left(\rho_{\mathrm{L}}\left(F^W(\pi, m, n; h)^*\right)\rho_{\mathrm{L}}\left(F^W(\pi', m', n'; h')\right)\right) = d_\pi^{-1}\delta_{\pi,\pi'}\delta_{m,m'}\delta_{n,n'}\delta_h(h'). \tag{3.36}$$

Proof:

The first statement is a consequence of the adjointness property (2.) in proposition III.2:

$$\begin{aligned}
F^{W*}_\sigma(h,g) &= \overline{F^W_\sigma(h^{-1}, h^{-1}g)} = \overline{F^R_\sigma(h^{-1}, \sqrt{h}h^{-1}, g)} = \overline{F^R_\sigma(h^{-1}, \sqrt{h}^{-1}g)} \\
&= \sum_{\pi\in\hat{G}} d_\pi \overline{\mathrm{tr}\left(\pi(h)\sigma(\pi, \sqrt{h}^{-1}g)\right)} \\
&= \sum_{\pi\in\hat{G}} d_\pi\, \mathrm{tr}\left(\pi(h)^*\sigma(\pi, \sqrt{h}^{-1}g)^*\right) \\
&= F^R_{\sigma^*}(h, \sqrt{h}^{-1}g) = F^W_{\sigma^*}(h,g).
\end{aligned} \tag{3.37}$$

The second statement follows from the orthogonality relations (3.18) and (3.34):

$$\begin{aligned}
&\mathrm{tr}\left(\rho_{\mathrm{L}}\left(F^W(\pi, m, n; h)^*\right)\rho_{\mathrm{L}}\left(F^W(\pi', m', n'; h')\right)\right) \\
&= \sum_{k=1}^{d_\pi}\sum_{k'=1}^{d_{\pi'}} \overline{\pi(\sqrt{h}^{-1})_{mk}}\pi'(\sqrt{h'}^{-1})_{m'k'}\, \mathrm{tr}\left(\rho_{\mathrm{L}}\left(F^R(\pi, k, n; h)^*\right)\rho_{\mathrm{L}}\left(F^R(\pi', k', n'; h')\right)\right) \\
&= d_\pi^{-1}\delta_{\pi,\pi'}\delta_{n,n'}\delta_h(h')\sum_{k=1}^{d_\pi}\pi(\sqrt{h})_{km}\pi(\sqrt{h}^{-1})_{m'k} = d_\pi^{-1}\delta_{\pi,\pi'}\delta_{m,m'}\delta_{n,n'}\delta_h(h'),
\end{aligned} \tag{3.38}$$

where the last line makes sense because $\sqrt{\ .\ } : G \to G$ admits a unique pointwise evaluation. □

Remark III.5:

Unfortunately, the definition of Fourier-Weyl elements seems to be problematic, as can be seen from a (formal) calculation:

$$\begin{aligned}
\hat{F}^W(g; \pi, m, n)(h, g') &= \sum_{\pi'\in\hat{G}} d_{\pi'}\sum_{m',n'=1}^{d_{\pi'}} \overline{\pi'(g)_{m'n'}}\int_G dh'\ \overline{\pi(h')_{mn}}F^W(\pi', m', n'; h')(h, g') \\
&= \overline{\pi(h)_{mn}}\delta_{\sqrt{h}g}(g')
\end{aligned} \tag{3.39}$$

From the last line, we infer that the definition of $\hat{F}^W(g; \pi, m, n)$ would require the composition of δ with $\sqrt{\ .\ }$, which is not necessarily well-defined, because $\sqrt{\ .\ }$ is in general not even continuous.

Still, we can define a dequantisation map, if we restrict ourselves to operators $A : C^\infty(G) \to \mathscr{D}'(G)$, s.t. $\alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_A)(h,g) = F_A(h, \sqrt{h}g)$ is in $\mathscr{D}'(G \times G)$, in analogy with definition III.3.

Definition III.6:

The Weyl symbol of an operator $A : C^\infty \to \mathscr{D}'(G)$, s.t. $\alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_A)(h,g) = F_A(h, \sqrt{h}g)$ is in $\mathscr{D}'(G \times G)$, is

$$\sigma_A^W(\pi, g) := \mathscr{F}_1\left[\alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_A)\right](\pi, g) = \int_G dh\ \alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_A)(h,g)\pi(h). \tag{3.40}$$

From the definitions it is obvious, that we have

Corollary III.7:

The Weyl quantisation $\sigma \mapsto F_\sigma^W$ and the Weyl symbol map $A \mapsto \sigma_A^W$ are inverse to each other.

In accordance with proposition III.2, we collect some properties of the Weyl (de)quantisation, although, at this point, we refrain from specifying the sets of operators or symbols for which the procedure is well-defined any further. We will cure this in the following subsection, where we define the local calculi.

Proposition III.8:

Given continuous operators $A, B : C^\infty(G) \to \mathscr{D}'(G)$, s.t. the Weyl symbols $\sigma_A^W, \sigma_B^W \in \hat{\mathscr{D}}'(\hat{G} \times G)$ are well-defined, then we have

1. $\sigma^W_{U_h A U_h^*}(\pi, g) = \pi(h)\sigma^W_{(\mathrm{id} \times L_{h^{-1}})^* F_A}(\pi, g)\pi(h)^*$.
2. $\sigma^W_{U_h^R A U_h^{R*}}(\pi, g) = \sigma_A^W(\pi, gh)$.
3. *If the product AB and its Weyl symbol σ_{AB}^W are well-defined, we have*

$$\begin{aligned}\sigma_{AB}^W(\pi, g) = \sum_{\pi', \pi'' \in \hat{G}} d_{\pi'} d_{\pi''} \int_G dg' \int_G dg'' \pi(g')\pi(g'')\, &\mathrm{tr}(\pi'(g')^* \sigma_A^W(\pi', \sqrt{g'^{-1}}\sqrt{g'g''}g)) \\ &\times \mathrm{tr}(\pi''(g'')^* \sigma_B(\pi'', \sqrt{g''^{-1}}g'^{-1}\sqrt{g'g''}g)).\end{aligned} \tag{3.41}$$

<u>Proof:</u>

The properties 1.-3. are proved in a completely analogous way as the corresponding properties in proposition III.2.

□

Remark III.9:

The Weyl symbols of the elementary operators appearing in (3.1) equal their Kohn-Nirenberg symbols, since we have $F_{P_X}(h,g) = -i\varepsilon(R_X\delta_e)(h),\ X \in \mathfrak{g}$, and $F_f(h,g) = \delta_e(h)f(g),\ f \in C^\infty(G)$:

$$\begin{aligned} & \alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_f)(h,g) = \delta_e(h)f(\sqrt{h}g) = \delta_e(h)f(g) \\ \Rightarrow\quad & \sigma_f^W(\pi, g) = f(g)\mathbb{1}_{V_\pi} \\ & \alpha_{\mathrm{L}}^{-\frac{1}{2}}(F_{P_X})(h,g) = -i\varepsilon(R_X\delta_e)(h) \\ \Rightarrow\quad & \sigma_{P_X}^W(\pi, g) = i\varepsilon d\pi(X), \end{aligned} \tag{3.42}$$

where there first line makes sense, because $\sqrt{\,.\,} : G \to G$ is uniquely defined everywhere, although it is not continuous.

Remark III.10:
If we choose $G = \mathbb{R}^n$, and therefore $\hat{G} = \mathbb{R}^n$, and $\sigma_A, \sigma_B \in \mathscr{S}(\mathbb{R}^{2n})$, we can make still make sense out of the product formula (3.41), and a simple calculation shows that it is equivalent to the twisted product (3.7):

$$\begin{aligned}
\sigma^W_{AB}(p,x) \quad &= \quad \int_{\mathbb{R}^{2n}} \frac{dp'\ dx'}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \frac{dp''\ dx''}{(2\pi)^n} e^{ip\cdot(x'+x'')} e^{-ip'\cdot x'} e^{-ip''\cdot x''} \sigma^W_A(p', \tfrac{1}{2}x'' + x) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \sigma^W_B(p'', -\tfrac{1}{2}x' + x) \\
&\overset{\substack{x' \mapsto -2(x'-x) \\ x'' \mapsto 2(x''-x)}}{=} \int_{\mathbb{R}^{2n}} \frac{dp'\ dx'}{(4\pi)^n} \int_{\mathbb{R}^{2n}} \frac{dp''\ dx''}{(4\pi)^n} e^{-2i(p-p')\cdot x'} e^{2i(p-p'')\cdot x''} e^{2i(p-p')\cdot x} e^{-2i(p-p'')\cdot x} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \sigma^W_A(p', x'') \sigma^W_B(p'', x') \\
&= \quad \int_{\mathbb{R}^{2n}} \frac{dp'\ dx'}{(4\pi)^n} \int_{\mathbb{R}^{2n}} \frac{dp''\ dx''}{(4\pi)^n} e^{2i((p-p')\cdot(x-x') - (p-p'')\cdot(x-x''))} \sigma^W_A(p', x'') \sigma^W_B(p'', x'),
\end{aligned} \tag{3.43}$$

where we recognize the last line as an alternative formula for the twisted product (3.7) (cf.[11]).

So far, we have not dealt with the question of the existence of an asymptotic expansion for the symbol $\sigma^{(W)}_{AB}$ of an operator product $AB : C^\infty(G) \to C^\infty(G)$. This will be done in the next subsection, where we introduce an ε-dependent expansion of $\sigma^{(W)}_{AB}$ in the local calculi. In contrast, Ruzhansky and Turunen[5] define a global symbolic calculus, but the ε-dependence remains rather opaque in their setting. We will come back to this point in subsections III D.

2. *A local calculus of Paley-Wiener-Schwartz symbols*

Following Rieffel and Landsman[7,28,29], we localise the quantisations discussed in the previous subsection in the sense, that we pass from global symbols $\sigma \in \hat{\mathscr{D}}'(\hat{G} \times G)$, living on G and its unitary dual $\hat{G}$, to local symbols $\sigma \in \mathscr{D}'(T^*G)$, living on $T^*G \cong G \times \mathfrak{g}^*$ (by right translation), via the exponential map $\exp : \mathfrak{g} \to G$. More precisely, this works as follows: exp defines a diffeomorphism between an open neighbourhood $U \subset \mathfrak{g}$ of $0 \in \mathfrak{g}$ (possibly Ad-invariant) and an open neighbourhood $V \subset G$ of $e \in G$[42]. Additionally, we define Fourier transform and its inverse between functions on $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$\begin{aligned}
\mathscr{F}[F](\theta) = \hat{F}(\theta) &:= \int_{\mathfrak{g}} dX\ e^{-i\theta(X)} F(X),\ \ F \in L^1(\mathfrak{g}), \theta \in \mathfrak{g}^* \\
\mathscr{F}^{-1}[\sigma](X) = \check{\sigma}(X) &:= \int_{\mathfrak{g}^*} \frac{d\theta}{(2\pi)^n} e^{i\theta(X)} \sigma(\theta),\ \ \sigma \in L^1(\mathfrak{g}^*), X \in \mathfrak{g},
\end{aligned} \tag{3.44}$$

where we fix the normalisation of the Lebesque measures dX and $d\theta$ via the exponential map and the Haar measure dg on G (cp. section II):

$$\int_V dg\ f(g) = \int_U dX\ j(X)^2 f(\exp(X)),\ \ f \in C_c(V). \tag{3.45}$$

Here, j is the analytic function $j(H) = \prod_{\alpha \in R^+} \frac{\sin(\alpha(H))}{\alpha(H)},\ H \in \mathfrak{t} \subset \mathfrak{g}$, in the notation of section II (see especially (2.20) and (2.21)). Now, we are in a position to make the

Definition III.11:

Given a function $\sigma \in C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G)^{43}$ s.t. $\check{\sigma}^1$, the inverse Fourier transform of σ in the first variable, is in $\mathscr{D}(\mathfrak{g})\hat{\otimes}\, C^\infty(G)$ with $\mathrm{supp}_1(\check{\sigma}^1) \subset U$, we define $F_\sigma \in C^\infty(G)\hat{\otimes}\, C^\infty(G)$ by:

$$F_\sigma(h,g) := \check{\sigma}^1(X_h,g) = \int_{\mathfrak{g}^*} \frac{d\theta}{(2\pi)^n} e^{i\theta(X_h)}\sigma(\theta,g) \text{ for } X_h := \exp^{-1}(h), \tag{3.46}$$

which is well-defined due to the support properties of $\check{\sigma}^1$. We also define an ε-scaled version of (3.46)*:*

$$\forall \epsilon \in (0,1]: \ F^\varepsilon_\sigma(h,g) := \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_h,g) = \int_{\mathfrak{g}^*} \frac{d\theta}{(2\pi\varepsilon)^n} e^{\frac{i}{\varepsilon}\theta(X_h)}\sigma(\theta,g). \tag{3.47}$$

We call $F^{(\varepsilon)}_\sigma$ the (ε-scaled) Kohn-Nirenberg quantisation *of σ, as it defines a compact operator on $L^2(G)$ via the left integrated representation $\rho_{\mathrm{L}} : C(G) \rtimes_{\mathrm{L}} G \to \mathscr{K}(L^2(G))$. Clearly, the ε-scaled quantisation extends to those σ, s.t. $\mathrm{supp}_1(\check{\sigma}^1) \subset \varepsilon^{-1}U =: U_\varepsilon$.*

By standard distributional reasoning, the quantisation extends to distributions $\sigma \in \hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, \mathscr{D}'(G)$, i.e. $\check{\sigma}^1 \in \mathscr{E}'_U(\mathfrak{g})\hat{\otimes}\, \mathscr{D}'(G)$, where $\mathscr{E}'_U(\mathfrak{g})$ is the space of compactly supported distributions in $U \subset \mathfrak{g}$.

Furthermore, if we define the (smooth) square root $\sqrt{\,.\,} : V \subset G \to V \subset G$ by $\sqrt{g} = \exp(\frac{1}{2}X_g)$, we can deform the Kohn-Nirenberg quantisation to a Weyl quantisation in analogy with the previous subsection:

$$F^{W,\varepsilon}_\sigma(h,g) := \alpha_{\mathrm{L}}^{\frac{1}{2}}(F^\varepsilon_\sigma)(h,g) = F^\varepsilon_\sigma(h,\sqrt{h^{-1}}g). \tag{3.48}$$

The Weyl quantisation will, in general, not be well-defined for $\sigma \in \hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, \mathscr{D}'(G)$, but surely for $\sigma \in \hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G)$.

Corollary III.12:

The Kohn-Nirenberg and Weyl quantisation have following adjointness (in the sense of (2.6)*) and covariance (w.r.t. to G) properties (cp. proposition III.2 & III.8):*

1. $\left(F^{(\varepsilon)}_\sigma\right)^* = \alpha_{\mathrm{L}}(F^{(\varepsilon)}_{\overline{\sigma}})$ *and* $\left(F^{W,(\varepsilon)}_\sigma\right)^* = F^{W,(\varepsilon)}_{\overline{\sigma}}$,
2. $U_h F^{(\varepsilon)}_\sigma U^*_h = \alpha_{\mathrm{L}}(h)\left(F^{(\varepsilon)}_{\left(Ad^*_{h^{-1}}\right)^*\sigma}\right)$ & $U_h F^{W,(\varepsilon)}_\sigma U^*_h = \alpha_{\mathrm{L}}\left(h\sqrt{h^{-1}(\,.\,)h}\sqrt{(\,.\,)^{-1}}\right)\left(F^{W,(\varepsilon)}_{\left(Ad^*_{h^{-1}}\right)^*\sigma}\right)$,
3. $U^R_h F^{(\varepsilon)}_\sigma U^{R*}_h = \alpha_{\mathrm{R}}(h)\left(F^{(\varepsilon)}_\sigma\right) = F^{(\varepsilon)}_{\left(R^*_{g^{-1}}\right)^*\sigma}$ & $U^R_h F^{W,(\varepsilon)}_\sigma U^{R*}_h = \alpha_{\mathrm{R}}(h)\left(F^{W,(\varepsilon)}_\sigma\right) = F^{W,(\varepsilon)}_{\left(R^*_{g^{-1}}\right)^*\sigma}$,

where $\sigma \in \hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, \mathscr{D}'(G)$, $h \in G$, α_{R} is the right regular representation of G, R^* is the pullback (right) action on T^*G and $Ad^* : G \to GL(\mathfrak{g}^*)$ denotes the coadjoint action.

Proof:

We prove the statements for $\sigma \in C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes} C^\infty(G)$, which implies them by standard distributional reasoning for $\sigma \in \hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, \mathscr{D}'(G)$ or $\hat{\mathscr{E}'}_U(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G)$.

1.

$$\left(F^\varepsilon_\sigma\right)^*(h,g) = \overline{F^\varepsilon_\sigma(h^{-1},h^{-1}g)} = \overline{\varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h^{-1}},h^{-1}g)} \tag{3.49}$$

$$= \overline{\varepsilon^{-n}\check{\sigma}^1(-\varepsilon^{-1}X_h, h^{-1}g)} = \varepsilon^{-n}\check{\overline{\sigma}}^1(\varepsilon^{-1}X_h, h^{-1}g) = \alpha_{\mathrm{L}}(F^{\varepsilon}_{\overline{\sigma}})(h,g).$$

Here, the third equality follows from $h^{-1} = \exp(X_h)^{-1} = \exp(-X_h)$, while the fourth equality follows from the interplay of the inverse Fourier transform and complex conjugation. By the same reasoning, we find:

$$\begin{aligned}\left(F^{W,\varepsilon}_{\sigma}\right)^*(h,g) &= \overline{F^{\varepsilon}_{\sigma}(h^{-1}, \sqrt{h}h^{-1}g)} = \overline{\varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h^{-1}}, \sqrt{h^{-1}}g)} \\ &= \overline{\varepsilon^{-n}\check{\sigma}^1(-\varepsilon^{-1}X_h, \sqrt{h^{-1}}g)} = \varepsilon^{-n}\check{\overline{\sigma}}^1(\varepsilon^{-1}X_h, \sqrt{h^{-1}}g) \\ &= F^{W,\varepsilon}_{\overline{\sigma}}(h,g).\end{aligned} \tag{3.50}$$

2.

$$\begin{aligned}(U_h F^{\varepsilon}_{\sigma} U^*_h)(h', g) &= F^{\varepsilon}_{\sigma}(h^{-1}h'h, h^{-1}g) = \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h^{-1}h'h}, h^{-1}g) \\ &= \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}Ad_{h^{-1}}(X_{h'}), h^{-1}g) \\ &= \varepsilon^{-n}\mathscr{F}_1^{-1}[(Ad^*_{h^{-1}})^*\sigma](\varepsilon^{-1}X_{h'}, h^{-1}g) \\ &= \alpha_{\mathrm{L}}(h)\left(F^{\varepsilon}_{\left(Ad^*_{h^{-1}}\right)^*\sigma}\right)(h', g).\end{aligned} \tag{3.51}$$

The third equality follows from the definition of the adjoint action $Ad: G \to GL(\mathfrak{g})$, and the fourth equality follows from the Ad-invariance of the Lebesgue measure $d\theta$ and the definition of the coadjoint action. Analogously, we find:

$$\begin{aligned}(U_h F^{W,\varepsilon}_{\sigma} U^*_h)(h', g) &= F^{\varepsilon}_{\sigma}(h^{-1}h'h, \sqrt{(h^{-1}h'h)^{-1}}h^{-1}g) \\ &= \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h^{-1}h'h}, \sqrt{h'^{-1}}\sqrt{h'}\sqrt{(h^{-1}h'h)^{-1}}h^{-1}g) \\ &= \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}Ad_{h^{-1}}(X_{h'}), \sqrt{h'^{-1}}\sqrt{h'}\sqrt{(h^{-1}h'h)^{-1}}h^{-1}g) \\ &= \varepsilon^{-n}\mathscr{F}_1^{-1}[(Ad^*_{h^{-1}})^*\sigma](\varepsilon^{-1}X_{h'}, \sqrt{h'^{-1}}\sqrt{h'}\sqrt{(h^{-1}h'h)^{-1}}h^{-1}g) \\ &= \alpha_{\mathrm{L}}(h\sqrt{h^{-1}h'h}\sqrt{h'^{-1}})\left(F^{W,\varepsilon}_{\left(Ad^*_{h^{-1}}\right)^*\sigma}\right)(h', g).\end{aligned} \tag{3.52}$$

□

3. By commutativity of left and right action, it follows:

$$(U_h F^{\varepsilon}_{\sigma} U^*_h)(h', g) = F^{\varepsilon}_{\sigma}(h', gh) = \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h'}, gh) = F^{\varepsilon}_{\left(R^*_{g^{-1}}\right)^*\sigma}, \tag{3.53}$$

$$(U_h F^{W,\varepsilon}_{\sigma} U^*_h)(h', g) = F^{\varepsilon}_{\sigma}(h', \sqrt{h'^{-1}}gh) = \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{h'}, \sqrt{h'^{-1}}gh) = F^{W,\varepsilon}_{\left(R^*_{g^{-1}}\right)^*\sigma}. \tag{3.54}$$

Our definition of the Weyl quantisation is indeed equivalent to the one given by Landsman in terms of a "geodesic midpoint construction" (cf.[7], Definition II.3.4.4.)

Lemma III.13:

The operator defined by the Weyl quantisation $F^{W,\varepsilon}_{\sigma}$ of $\sigma \in C^{\infty}_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes}C^{\infty}(G)$ is equivalent to the

operator defined by the Weyl kernel

$$K_\sigma^{W,(\varepsilon)}(h,g) = \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}\nu_\delta^{-1}(h,g)), \tag{3.55}$$

where $\nu_\delta^{-1} : V \times V \to TV \cong V \times \mathfrak{g}$ *maps* (h,g) *to the tangent vector at the midpoint of the geodesic from* h *to* g *(w.r.t. an invariant metric on* G*).*

Proof:

By definition the operator corresponding to the Weyl quantisation $F_\sigma^{W,\varepsilon}$ is:

$$\forall \Psi \in L^2(G): \ \left(\rho_{\mathrm{L}}\left(F_\sigma^{W,\varepsilon}\right)\Psi\right)(g) = \int_G dh\ F_\sigma^{W,\varepsilon}(h,g)\Psi(h^{-1}g) = \int_G dh\ F_\sigma^{W,\varepsilon}(gh^{-1},g)\Psi(h). \tag{3.56}$$

Thus, the kernel of $\rho_{\mathrm{L}}\left(F_\sigma^{W,\varepsilon}\right)$ is

$$\begin{aligned} K_{\rho_{\mathrm{L}}\left(F_\sigma^{W,\varepsilon}\right)}(h,g) &= F_\sigma^{W,\varepsilon}(gh^{-1},g) \\ &= \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{gh^{-1}}, \sqrt{gh^{-1}}^{-1} g) \\ &= \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}X_{gh^{-1}}, \exp(-\tfrac{1}{2}X_{gh^{-1}})g). \end{aligned} \tag{3.57}$$

□

But, $(\exp(-\frac{1}{2}X_{gh^{-1}})g, X_{gh^{-1}})$ is exactly the point in $V \times \mathfrak{g}$ corresponding to the tangent vector at the midpoint of the geodesic $\gamma_{h\to g} : [0,1] \to V,\ \gamma_{h\to g}(t) = \exp(tX_{gh^{-1}})h$, under right translation, because $\exp(\frac{1}{2}X_{gh^{-1}})h = \exp((1-\frac{1}{2})X_{gh^{-1}})h = \exp(-\frac{1}{2}X_{gh^{-1}})\exp(X_{gh^{-1}})h = \exp(-\frac{1}{2}X_{gh^{-1}})g$. We conclude:

$$K_{\rho_{\mathrm{L}}\left(F_\sigma^{W,\varepsilon}\right)}(h,g) = \varepsilon^{-n}\check{\sigma}^1(\varepsilon^{-1}\nu_\delta^{-1}(h,g)) = K_\sigma^{W,(\varepsilon)}(h,g). \tag{3.58}$$

Therefore, we have the following theorem proven by Landsman in the context of Riemannian manifolds.

Theorem III.14 (cf.[7], Theorem II.3.5.1. & Theorem III.2.8.1):

The composition of ρ_{L} *and the Weyl quantisation*

$$Q_\varepsilon^W := \rho_{\mathrm{L}} \circ F_{(\ .\ )}^{W,\varepsilon} : C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G) \to \mathscr{K}(L^2(G)) \tag{3.59}$$

is a nondegenerate strict quantization of $C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G) \subset C_0(G\times\mathfrak{g}^*) \cong C_0(T^*G)$ *on* $\varepsilon \in (0,1]$*, i.e. we have for all* $\sigma,\tau \in C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*,\mathbb{R})\hat{\otimes}\, C^\infty(G,\mathbb{R})$*:*

1. *(nondegeneracy):*
   $\forall \epsilon \in (0,1]: \ Q_\varepsilon^W(\sigma) = 0 \Leftrightarrow \sigma = 0.$
2. *(Rieffel's condition):*
   $\varepsilon \mapsto ||Q_\varepsilon^W(\sigma)||$ *is continuous on* $(0,1]$*, especially* $\lim_{\varepsilon\to 0}||Q_\varepsilon^W(\sigma)|| = ||\sigma||_\infty$.
3. *(von Neumann's condition):*
   $\lim_{\varepsilon\to 0}\left|\left|\frac{1}{2}\left(Q_\varepsilon^W(\sigma)Q_\varepsilon^W(\tau) + Q_\varepsilon^W(\tau)Q_\varepsilon^W(\sigma)\right) - Q_\varepsilon^W(\sigma\tau)\right|\right| = 0.$
4. *(Dirac's condition):*
   $\lim_{\varepsilon\to 0}\left|\left|\frac{i}{\varepsilon}\left[Q_\varepsilon^W(\sigma), Q_\varepsilon^W(\tau)\right] - Q_\varepsilon^W\left(\{\sigma,\tau\}_{T^*G}\right)\right|\right| = 0.$

Here, $\{\ ,\ \}_{T^*G}$ is the canonical Poisson structure on T^*G, which takes the following for on $G\times\mathfrak{g}^*$ by right translation (cp.[7], Proposition III.1.4.1):

$$\forall\sigma,\tau\in C^\infty(T^*G):\ \{\sigma,\tau\}_{T^*G}=\langle\partial_\theta\sigma,R\tau\rangle-\langle R\sigma,\partial_\theta\tau\rangle+\{\sigma,\tau\}_-,\tag{3.60}$$

where $(\langle X,Rf\rangle)(g)=(R_Xf)(g)=\frac{d}{dt}_{|t=0}f(\exp(tX)g)$ is the right differential on G, and $\{f,f'\}_-(\theta)=-\theta([\partial_\theta f,\partial_\theta f'])$ is the (minus) Lie-Poisson structure on $\mathfrak{g}^*$.

Remark III.15:
For $\sigma\in C^\infty_{\mathrm{PW},U}(\mathfrak{g}^*)\hat{\otimes}C^\infty(G)$, we have $K_{\rho_{\mathrm{L}}(F^{(W),\varepsilon}_\sigma)}\in C^\infty(G)\hat{\otimes}C^\infty(G)\cong C^\infty(G\times G)$ by definition. Therefore, $Q^{(W)}_\varepsilon(\sigma)$ is not only a compact operator on $L^2(G)$, but preserves $C^\infty(G)$ and extends to a smoothing operator from $\mathscr{D}'(G)$ to $C^\infty(G)$ by the properties of convolution (cf.[44], Theorem 4.1.1.).

Before we introduce the Paley-Wiener-Schwartz symbol spaces $S^{K,m}_{\mathrm{PW},\rho,\delta}\subset\hat{\mathscr{E}}'_U(\mathfrak{g}^*)\hat{\otimes}C^\infty(G)$, we discuss the quantization of symbols that are polynomial in the momentum variables $\theta\in\mathfrak{g}^*$. To make this precise, we recall that the left and right pullback actions of G on T^*G are strongly Hamiltonian and compatible, i.e. we have bi-equivariant Poisson momentum maps (cf.[7], section III.1.4):

$$\begin{aligned}
J^{L^*_{(\ .\ )^{-1}}}:&T^*(G)\to\mathfrak{g}^*, \\
J^{L^*_{(\ .\ )^{-1}}}(\theta,g)&=\theta, & J^{L^*_{(\ .\ )^{-1}}}(L^*_{h^{-1}}(\theta,g))&=Ad^*_h\left(J^{L^*_{(\ .\ )^{-1}}}(\theta,g)\right), \\
\{J^{L^*_{(\ .\ )^{-1}}}_X,J^{L^*_{(\ .\ )^{-1}}}_Y\}_{T^*G}&=-J^{L^*_{(\ .\ )^{-1}}}_{[X,Y]}, & J^{L^*_{(\ .\ )^{-1}}}(R^*_{h^{-1}}(\theta,g))&=J^{L^*_{(\ .\ )^{-1}}}(\theta,g), \\
J^{R^*_{(\ .\ )^{-1}}}:&T^*(G)\to\mathfrak{g}^*, \\
J^{R^*_{(\ .\ )^{-1}}}(\theta,g)&=-Ad^*_{g^{-1}}(\theta), & J^{R^*_{(\ .\ )^{-1}}}(R^*_{h^{-1}}(\theta,g))&=Ad^*_h\left(J^{R^*_{(\ .\ )^{-1}}}(\theta,g)\right), \\
\{J^{R^*_{(\ .\ )^{-1}}}_X,J^{R^*_{(\ .\ )^{-1}}}_Y\}_{T^*G}&=-J^{R^*_{(\ .\ )^{-1}}}_{[X,Y]}, & J^{R^*_{(\ .\ )^{-1}}}(L^*_{h^{-1}}(\theta,g))&=J^{R^*_{(\ .\ )^{-1}}}(\theta,g),
\end{aligned}\tag{3.61}$$

$$\{J^{L^*_{(\ .\ )^{-1}}}_X,J^{R^*_{(\ .\ )^{-1}}}_Y\}_{T^*G}=0,\qquad X,Y\in\mathfrak{g}\tag{3.62}$$

where we identified $T^*G\cong G\times\mathfrak{g}^*$ by right translation, as above, and defined $J^\bullet_X(\theta,g):=J^\bullet(\theta,g)(X),\ X\in\mathfrak{g}$. This allows us, to make the notion of polynomial symbols precise:

Definition III.16:
Given a (smooth) function σ on $T^*G\cong G\times\mathfrak{g}^*$ with values in multilinear maps from $\mathbb{R}\oplus\bigoplus_{n=1}^N\mathfrak{g}^{\oplus n}$ to $\mathbb{R}$, $N\in\mathbb{N}_0$, of the form

$$\begin{aligned}
\sigma(\theta,g)(\oplus X)&=f_0(g)+\sum_{n=1}^N\sum_{i_1,\dots,i_n}f_{i_1\dots i_n}(g)J^{L^*_{(\ .\ )^{-1}}}_{X_{i_1}}(\theta,g)\dots J^{L^*_{(\ .\ )^{-1}}}_{X_{i_n}}(\theta,g)\\
&=f_0(g)+\sum_{n=1}^N\sum_{i_1,\dots,i_n}f_{i_1\dots i_n}(g)\theta(X_{i_1})\theta(X_{i_n})
\end{aligned}\tag{3.63}$$

for $\oplus X \in \bigoplus_{n=1}^{N} \mathfrak{g}^{\oplus n}$, $\{f_0, f_{i_1\dots i_n}\}_{i_1,\dots,i_n} \subset \mathscr{D}'(G)$*, we call each* $\sigma(\oplus X) \in \hat{\mathscr{E}}'_U(\mathfrak{g}^*)\hat{\otimes}\,\mathscr{D}'(G)$ *a* polynomial symbol of degree $\leq N$*. If* $\sigma(\oplus X) \in \hat{\mathscr{E}}'_U(\mathfrak{g}^*)\hat{\otimes}\, C^\infty(G)$*, we say that* σ *is smooth.*

The Paley-Wiener-Schwartz theorem[44] allows us to characterise the quantisation of polynomial symbols:

Corollary III.17:

For $N \in \mathbb{N}_0$*:* σ *is a polynomial symbol of degree* $\leq N \Leftrightarrow \check{\sigma}^1$ *is a distribution of order* $\leq N$ *in* $\mathfrak{g}$ *with values in* $\mathscr{D}'(G)$ *and* $\operatorname{supp}_1(\check{\sigma}^1) \subset \{0\} \Leftrightarrow F_\sigma^{(\varepsilon)}$ *is a distribution of order* $\leq N$ *in* G *with values in* $\mathscr{D}'(G)$ *and* $\operatorname{supp}_1(F_\sigma^{(\varepsilon)}) \subset \{e\}$*.*

Among the polynomial symbols, we find the special cases $\sigma_f(\theta,g) := f(g)$, $f \in C^\infty(G)$, and $\sigma_X(\theta,g) := J_X^{L^*_{(\ .\ )^{-1}}}(\theta,g)$, $X \in \mathfrak{g}$,. After a moments reflection, we see that the quantisation of these symbols gives rise to the commutation relations (3.1):

$$\forall \Psi \in C^\infty(G): \ \left(Q_\varepsilon^{(W)}(\sigma_f)\Psi\right)(g) = f(g)\Psi(g), \qquad \left(Q_\varepsilon^{(W)}(\sigma_X)\Psi\right)(g) = -i\varepsilon(R_X\Psi)(G), \tag{3.64}$$

$$\begin{aligned} Q_\varepsilon^{(W)}(\{\sigma_f,\sigma_{f'}\}_{T^*G}) &= \tfrac{i}{\varepsilon}[Q_\varepsilon^{(W)}(\sigma_f), Q_\varepsilon^{(W)}(\sigma_{f'})] = 0, \\ Q_\varepsilon^{(W)}(\{\sigma_X,\sigma_f\}_{T^*G}) &= \tfrac{i}{\varepsilon}[Q_\varepsilon^{(W)}(\sigma_X), Q_\varepsilon^{(W)}(\sigma_f)] = R_X f, \\ Q_\varepsilon^{(W)}(\{\sigma_X,\sigma_Y\}_{T^*G}) &= \tfrac{i}{\varepsilon}[Q_\varepsilon^{(W)}(\sigma_X), Q_\varepsilon^{(W)}(\sigma_Y)] = i\varepsilon R_{[X,Y]}. \end{aligned} \tag{3.65}$$

Let us come to the definition of the Paley-Wiener-Schwartz symbol spaces $S^{K,m}_{\mathrm{PW},\rho,\delta}$, which are analogous to the (classical) symbol spaces $S^m_{\rho,\delta}$ in the theory of pseudo-differential operators and Weyl quantisation on $\mathbb{R}^n$ (cf.[11,37,44]). The main obstacle to such a definition is the fact that the exponential is no longer a diffeomorphism, which is why we need to deal with compactly supported instead of tempered distributions in $\mathfrak{g}$, and thus by the Paley-Wiener-Schwartz theorem with entire analytic functions on $\mathfrak{g}^*$ by means of the Fourier transform. Concerning the development of an asymptotic calculus, the analyticity requirement forbids the use of 0-excision functions, which are standard in pseudo-differential calculus. Luckily, this alleged shortcoming can be dealt with by the method of kernel cut-off operator from the theory of Volterra-Mellin pseudo-differential operators (cf.[32,33]).

Definition III.18:

Let $K \sqsubset \mathfrak{g}$ *be a convex compact subset and* $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$*. A function* $\sigma \in C^\infty(\mathfrak{g}^*_{\mathbb{C}}, C^\infty(G))$ *belongs to the space of* Paley-Wiener-Schwartz symbols $S^{K,m}_{\mathrm{PW},\rho,\delta}$ *if the following conditions are satisfied:*

1. $\sigma : \mathfrak{g}^*_{\mathbb{C}} \to C^\infty(G)$ *is (weakly) holomorphic*[45]*,*
2. $\forall \alpha,\beta \in \mathbb{N}_0^n : \exists C_{\alpha\beta} > 0 : \forall \theta \in \mathfrak{g}^*_{\mathbb{C}} :\ \sup_{g\in G} |(R^\alpha \partial_\theta^\beta \sigma)(\theta,g)| \leq C_{\alpha\beta}\langle\theta\rangle^{m-|\beta|\rho+|\alpha|\delta} e^{H_K(\Im(\theta))}$*,*

where $\langle\theta\rangle := (1+|\theta|^2_{\mathfrak{g}^*_{\mathbb{C}}})^{\frac{1}{2}}$ *is the* standard regularized distance*,* $H_K(\Im(\theta)) := \sup_{X\in K} \Im(\theta)(X)$ *is the* supporting function of K*, and we use the standard multi index notation (w.r.t. a fixed ordered basis* $\{\tau_i\}_{i=1}^n \subset \mathfrak{g}$ *and its dual in* $\mathfrak{g}^*$*). Clearly, the definition is independent of the ordering of the right multi-differentials* $R^\alpha = R_1^{\alpha_1}\dots R_n^{\alpha_n}$*, because the commutator* $[R_i,R_j] = f_{ij}^k R_k$ *reduces the order and* $\delta \geq 0$*.*
$S^{K,m}_{\mathrm{PW},\rho}(\mathfrak{g}^*_{\mathbb{C}})$ *denotes the analogue of* $S^{K,m}_{\mathrm{PW},\rho,\delta}$ *with* $C^\infty(G)$ *replaced by* $\mathbb{C}$*.*

Clearly, smooth polynomial symbols of degree $\leq N$ belong to $S^{K,N}_{\mathrm{PW},1,0}$ for all $K \sqsubset \mathfrak{g}$. The following relations among the symbol spaces are immediate consequences of the definition:

$$S^{K,m}_{\mathrm{PW},\rho,\delta} \subset S^{K,m}_{\mathrm{PW},\rho',\delta},\ \rho \geq \rho', \qquad S^{K,m}_{\mathrm{PW},\rho,\delta} \subset S^{K,m}_{\mathrm{PW},\rho,\delta'},\ \delta \leq \delta' \tag{3.66}$$
$$S^{K,m}_{\mathrm{PW},\rho,\delta} \subset S^{K',m}_{\mathrm{PW},\rho,\delta},\ K \subset K', \qquad S^{K,m}_{\mathrm{PW},\rho,\delta} \subset S^{K,m'}_{\mathrm{PW},\rho,\delta},\ m \leq m'.$$

This suggests the definition of the following spaces:

$$S^{K,\infty}_{\mathrm{PW},\rho,\delta} := \bigcup_{m\in\mathbb{R}} S^{K,m}_{\mathrm{PW},\rho,\delta}, \quad S^{K,-\infty}_{\mathrm{PW}} := \bigcap_{m\in\mathbb{R}} S^{K,m}_{\mathrm{PW},\rho,\delta}. \tag{3.67}$$

The Kohn-Nirenberg and Weyl quantisation of the restriction $\sigma_{|\,\mathfrak{g}^*}$ of $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$ to $\mathfrak{g}^* \subset \mathfrak{g}^*_{\mathbb{C}}$ define continuous operators on $C^\infty(G)$. In the following, we will abuse notation and suppress the restriction index.

Corollary III.19:
Given $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$, there exists $\varepsilon \in (0,1]$ s.t. $K \subset U_\varepsilon$. Then $F^{W,\varepsilon}_\sigma$ defines a continuous operator on $C^\infty(G)$ via ρ_{L}. Moreover, if $\sigma \in S^{K,-\infty}_{\mathrm{PW},\rho,\delta}$, then $\rho_{\mathrm{L}}\left(F^{W,\varepsilon}_\sigma\right)$ is a smoothing operator from $\mathscr{D}'(G)$ to $C^\infty(G)$.

Proof:
By the Paley-Wiener-Schwartz theorem, $\check{\sigma}^1$ is a distribution of order $\leq N$, for $N \in \mathbb{N}_0$ s.t. $m \leq N$, in $\mathfrak{g}$ with values in $C^\infty(G)$ and $\mathrm{supp}_1(\check{\sigma}^1) \subset K$, which implies the first statement. The second statement follows from remark III.15. □

The optimal constants $C_{\alpha\beta} > 0$ in definition III.18 turn the symbol spaces $S^{K,m}_{\mathrm{PW},\rho,\delta}$ into Fréchet spaces.

Proposition III.20:
Fix a convex compact subset $K \sqsubset \mathfrak{g}$ and $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$. The countable family of (ordered) seminorms

$$||\sigma||^{(K,m,\rho,\delta)}_k := \sup_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ |\alpha|+|\beta|\leq k}} \sup_{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}} \langle\theta\rangle^{-m+|\beta|\rho-|\alpha|\delta} e^{-H_K(\Im(\theta))} |(R^\alpha \partial^\beta_\theta \sigma)(\theta,g)|, \tag{3.68}$$

$k \in \mathbb{N}_0, \sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$, defines a Fréchet space topology on $S^{K,m}_{\mathrm{PW},\rho,\delta}$.

Proof:
We show that $S^{K,m}_{\mathrm{PW},\rho,\delta}$ is Hausdorff and complete in the locally convex topology defined by the seminorms. Since metrisability follows, because the family of seminorms is countable, we may conclude that the spaces are Fréchet.

1. Assuming $||\sigma||^{(K,m,\rho,\delta)}_k = 0$ for some $k \in \mathbb{N}_0$, we have $||\sigma||^{(K,m,\rho,\delta)}_0 = 0$, implying $u \equiv 0$. Therefore, $S^{K,m}_{\mathrm{PW},\rho,\delta}$ is separated, and thus Hausdorff.

2. Let us assume that $\{\sigma_i\}_{i=1}^\infty \subset S^{K,m}_{\mathrm{PW},\rho,\delta}$ is a Cauchy sequence, i.e. $\{\sigma_i\}_{i=1}^\infty$ is Cauchy for all seminorms $||\ .\ ||^{(K,m,\rho,\delta)}_k$, $k \in \mathbb{N}_0$. By the definition of the seminorms, we know that the

convergence of $\{R^\alpha\sigma_i\}_{i=1}^\infty$ is uniform on compact sets in $G\times\mathfrak{g}_\mathbb{C}^*$ for all $\alpha\in\mathbb{N}_0^n$, which implies the existence of limit $\sigma\in C^\infty(\mathfrak{g}_\mathbb{C}^*,C^\infty(G))$ in this sense. Similarly, we have compact convergence of $\{\Lambda(R^\alpha\sigma_i)\}_{i=1}^\infty$ for all $\Lambda\in\mathscr{E}'_\beta(G),\ \alpha\in\mathbb{N}_0^n$, where $\mathscr{E}'_\beta(G)$ is the dual of $C^\infty(G)$ with its strong topology (cf.[46]), which implies the existence (holomorphic) limits $\sigma^{\Lambda,\alpha}\in\mathscr{O}(\mathfrak{g}_\mathbb{C}^*)$ with linear dependence on $\Lambda\in\mathscr{E}'_\beta(G)$. Next, we show that the maps $\Lambda\mapsto\sigma^{\Lambda,\alpha}(\theta),\ \theta\in\mathfrak{g}_\mathbb{C}^*$, are bounded, which allows us to conclude that there exists $\sigma^\alpha(\theta)\in C^\infty(G)$ s.t. $\sigma^{\Lambda,\alpha}(\theta)=\Lambda(\sigma^\alpha(\theta))$, because $\mathscr{E}'_\beta(G)$ is Fréchet-Montel, and thus reflexive:

$$\begin{aligned}\forall\theta\in\mathfrak{g}_\mathbb{C}^*:\forall\epsilon>0\exists i_0\in\mathbb{N}:\forall i\geq i_0:\ |\sigma^{\Lambda,\alpha}(\theta)|&\leq|\sigma^{\Lambda,\alpha}(\theta)-\sigma_i^{\Lambda,\alpha}(\theta)|+|\sigma_i^{\Lambda,\alpha}(\theta)|\leq\epsilon+|\sigma_i^{\Lambda,\alpha}(\theta)|\\ &\leq\epsilon+\sup_{\substack{\sigma_k(\theta)\in\{\sigma_j(\theta)\}_{j=1}^\infty\\ k\in\mathbb{N}}}|\Lambda((R^\alpha\sigma_k)(\theta))|.\end{aligned}\tag{3.69}$$

As $\{(R^\alpha\sigma_j)(\theta)\}_{j=1}^\infty\subset C^\infty(G)$ is (weakly) bounded, the boundedness of the maps $\Lambda\mapsto\sigma^{\Lambda,\alpha}(\theta),\ \theta\in\mathfrak{g}_\mathbb{C}^*$, follows:

$$\begin{aligned}&\forall\Lambda\in\mathscr{E}'_\beta(G):\exists k_\Lambda\in\mathbb{N}_0,C_{\Lambda,\alpha}>0:\\ &\sup_{\substack{\sigma_i(\theta)\in\{\sigma_j(\theta)\}_{j=1}^\infty\\ i\in\mathbb{N}}}|\Lambda(\sigma_i^\alpha(\theta))|\leq C_{\Lambda,\alpha}\sup_{i\in\mathbb{N}}\sup_{\substack{g\in G\\|\gamma|\leq k_\Lambda}}|(R^{\alpha+\gamma}\sigma_i)(\theta,g)|\\ &\qquad\leq C_{\Lambda,\alpha}\langle\theta\rangle^{m+(k_\Lambda+|\alpha|)\delta}e^{H_K(\Im(\theta))}\sup_{i\in\mathbb{N}}||\sigma_i||_{k_\Lambda+|\alpha|}^{(K,m,\rho,\delta)}\\ &\qquad\leq\langle\theta\rangle^{m+k_\Lambda\delta}e^{H_K(\Im(\theta))}M_{k_\Lambda+|\alpha|}.\end{aligned}\tag{3.70}$$

By construction, the map $\theta\mapsto\sigma(\theta)$ is weakly holomorphic, and $\sigma^\alpha(\theta)=(R^\alpha\sigma)(\theta)$, because the above implies:

$$\forall\alpha\in\mathbb{N}_0^n,\theta\in\mathfrak{g}_\mathbb{C}^*:\forall\epsilon>0:\exists i_0\in\mathbb{N}:\forall i,j\geq i_0:\sup_{g\in G}|(R^\alpha\sigma_i)(\theta,g)-(R^\alpha\sigma)(\theta,g)|\quad<\epsilon\tag{3.71}$$

$$\begin{aligned}\Rightarrow\quad&\forall\Lambda\in\mathscr{E}'_\beta(G),\alpha\in\mathbb{N}_0^n,\theta\in\mathfrak{g}_\mathbb{C}^*:\forall\epsilon>0:\exists i_0\in\mathbb{N}:\forall i\geq i_0:\\ &|\Lambda(\sigma^\alpha(\theta))-\Lambda((R^\alpha\sigma)(\theta))|\leq|\Lambda(\sigma^\alpha(\theta))-\Lambda(\sigma_i^\alpha(\theta))|+|\Lambda((R^\alpha\sigma_i)(\theta))-\Lambda((R^\alpha\sigma)(\theta))|\\ &\qquad\leq|\Lambda(\sigma^\alpha(\theta))-\Lambda(\sigma_i^\alpha(\theta))|\\ &\qquad\quad+C_{\Lambda,\alpha}\sup_{\substack{g\in G\\|\gamma|\leq k_\Lambda}}|(R^\alpha\sigma_i)(\theta,g)-(R^\alpha\sigma)(\theta,g)|\\ &\qquad<\epsilon\end{aligned}\tag{3.72}$$

Furthermore, we know from the assumptions that the sequences

$$\{\tau_{\alpha,\beta,i}^{K,m,\rho,\delta}=\langle\ .\ \rangle^{-m+\rho|\beta|-\delta|\alpha|}e^{-H_K(\Im(\ .\ ))}(R^\alpha\partial_\theta^\beta\sigma_i)\}_{i=1}^\infty\subset C_b(\mathfrak{g}_\mathbb{C}^*\times G)\tag{3.73}$$

uniformly converge to limits $\tau^{K,m,\rho,\delta}_{\alpha,\beta} \in C_b(\mathfrak{g}^*_{\mathbb{C}} \times G)$. It remains to be concluded that:

$$\tau^{K,m,\rho,\delta}_{\alpha,\beta}(\theta,g) = \langle\theta\rangle^{-m+\rho|\beta|-\delta|\alpha|} e^{-H_K(\Im(\theta))}(R^\alpha \partial^\beta_\theta \sigma)(\theta,g), \tag{3.74}$$

which follows from the convergence properties established so far:

$$\begin{aligned}
\forall \alpha,\beta \in \mathbb{N}^n_0, (g,\theta) \in G\times\mathfrak{g}^*_{\mathbb{C}} : \forall\epsilon>0 : \exists i_0 \in \mathbb{N} : \forall i \geq i_0 : \\
&|\tau^{K,m,\rho,\delta}_{\alpha,\beta}(\theta,g) - \langle\theta\rangle^{-m+\rho|\beta|-\delta|\alpha|} e^{-H_K(\Im(\theta))}(R^\alpha \partial^\beta_\theta \sigma)(\theta,g)| \\
&\leq |\tau^{K,m,\rho,\delta}_{\alpha,\beta}(\theta,g) - \langle\theta\rangle^{-m+\rho|\beta|-\delta|\alpha|} e^{-H_K(\Im(\theta))}(R^\alpha \partial^\beta_\theta \sigma_i)(\theta,g)| \\
&\quad + \langle\theta\rangle^{-m+\rho|\beta|-\delta|\alpha|} e^{-H_K(\Im(\theta))}|(R^\alpha \partial^\beta_\theta \sigma_i)(\theta,g) - (R^\alpha \partial^\beta_\theta \sigma)(\theta,g)| \\
&< \epsilon
\end{aligned} \tag{3.75}$$

□

Remark III.21:
By the preceding proposition, we can give $S^{K,\infty}_{\mathrm{PW},\rho,\delta}$ a strict inductive limit topology (cf.[40]), which makes it an LF-space.

Lemma III.22:
For $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$, $\tau \in S^{K',m'}_{\mathrm{PW},\rho',\delta'}$ s.t. $\max(\delta,\delta') \leq \min(\rho,\rho')$, we have continuous maps:

1. $\forall \alpha,\beta \in \mathbb{N}^n_0 :\ \sigma \mapsto R^\alpha \partial^\beta_\theta \sigma \in S^{K,m-|\beta|\rho+|\alpha|\delta}_{\mathrm{PW},\rho,\delta}$.
2. $(\sigma,\tau) \mapsto \sigma\tau \in S^{K+K',m+m'}_{\mathrm{PW},\min(\rho,\rho'),\max(\delta,\delta')}$.

This implies that the Poisson bracket (3.60) *defines a bilinear operation*

$$\{\ ,\ \} : S^{K,m}_{\mathrm{PW},\rho,\delta} \times S^{K',m'}_{\mathrm{PW},\rho,\delta} \to S^{K+K',m+m'-\min(\rho-\delta,2\rho-1)}_{\mathrm{PW},\rho,\delta}. \tag{3.76}$$

<u>Proof:</u>

1. From the commutation relations $[R_i,R_j] = f^k_{ij}R_k$ and $\delta \geq 0$, we conclude:

$$\begin{aligned}
\sup_{g\in G}|(R^\gamma\partial^\epsilon_\theta(R^\alpha\partial^\beta_\theta\sigma))(\theta,g)| &\leq \sup_{g\in G}|(R^{\alpha+\gamma}\partial^{\beta+\epsilon}_\theta\sigma)(\theta,g)| \\
&\quad + \sum_{\substack{\epsilon\in\mathbb{N}^n_0 \\ |\zeta|<|\alpha|+|\gamma|}} \sup_{g\in G}|(R^\zeta\partial^{\beta+\epsilon}_\theta\sigma)(\theta,g)| \\
&\leq \langle\theta\rangle^{m-|\beta+\epsilon|\rho} e^{H_K(\Im(\theta))}\Bigg(C_{(\alpha+\gamma)(\beta+\epsilon)}\langle\theta\rangle^{|\alpha+\gamma|\delta} + \sum_{\substack{\epsilon\in\mathbb{N}^n_0 \\ |\zeta|<|\alpha|+|\gamma|}} C_{\zeta(\beta+\epsilon)}\langle\theta\rangle^{|\zeta|\delta}\Bigg) \\
&\leq C'_{(\alpha+\gamma)(\beta+\epsilon)}\langle\theta\rangle^{m-|\beta+\epsilon|\rho+|\alpha+\gamma|\delta} e^{H_K(\Im(\theta))}.
\end{aligned} \tag{3.77}$$

2. Using the Leibniz formula and the fact that $H_K + H_{K'} = H_{K+K'}$, we find:

$$\sup_{g\in G}|(R^\alpha\partial^\beta_\theta(\sigma\tau))(\theta,g)| \tag{3.78}$$

$$\leq \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma} \sum_{\substack{\epsilon\in\mathbb{N}_0^n\\ \epsilon\leq\beta}} \binom{\beta}{\epsilon} \sup_{g\in G} |(R^\gamma \partial_\theta^\epsilon \sigma)(\theta,g)| \sup_{g\in G} |(R^{\alpha-\gamma}\partial_\theta^{\beta-\epsilon}\tau)(\theta,g)|$$

$$\leq \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma} \sum_{\substack{\epsilon\in\mathbb{N}_0^n\\ \epsilon\leq\beta}} \binom{\beta}{\epsilon} C_{\gamma\epsilon} \langle\theta\rangle^{m-|\epsilon|\rho+|\gamma|\delta} e^{H_K(\Im(\theta))} C'_{(\alpha-\gamma)(\beta-\epsilon)} \langle\theta\rangle^{m'-|\beta-\epsilon|\rho'+|\alpha-\gamma|\delta'} e^{H_{K'}(\Im(\theta))}$$

$$\leq C''_{\alpha\beta} e^{H_K(\Im(\theta))+H_{K'}(\Im(\theta))} \langle\theta\rangle^{m+m'} \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\leq\alpha}} \binom{\alpha}{\gamma} \langle\theta\rangle^{|\gamma|\delta+|\alpha-\gamma|\delta'} \sum_{\substack{\epsilon\in\mathbb{N}_0^n\\ \epsilon\leq\beta}} \binom{\beta}{\epsilon} \langle\theta\rangle^{|\epsilon|\rho-|\beta-\epsilon|\rho'}$$

$$\leq C'''_{\alpha\beta} \langle\theta\rangle^{m+m'-|\beta|\min(\rho,\rho')+|\alpha|\max(\delta,\delta')} e^{H_{K+K'}(\Im(\theta))}. \qquad \square$$

As a preparation for the main theorem of this subsection, we define the kernel cut-off operator.

Definition III.23 (cp.[32], Definition 3.6):
For $\varphi \in C^\infty(\mathfrak{g})$, we define the kernel cut-off operator *by*

$$(H(\varphi)\sigma)(\theta,g) := \check{\sigma}^1(e^{-i\theta(\,\cdot\,)}\varphi) = \int_{\mathfrak{g}} dX\ e^{-i\theta(X)}\varphi(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{i\theta'(X)}\sigma(\theta,g) \tag{3.79}$$

for $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$, $(g,\theta) \in G\times\mathfrak{g}^*_{\mathbb{C}}$.

Remark III.24:
If $\theta \in \mathfrak{g}^* \subset \mathfrak{g}^*_{\mathbb{C}}$, we have

$$\begin{aligned} (H(\varphi)\sigma)(\theta,g) &= \check{\sigma}^1(e^{-i\theta(\,\cdot\,)}\varphi) = \check{\sigma}^1(e^{-i\theta(\,\cdot\,)}\varphi\chi) \\ &= \int_{\mathfrak{g}} dX\ e^{-i\theta(X)}\varphi(X)\chi(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{i\theta'(X)}\sigma(\theta,g) \\ &= \int_{\mathfrak{g}} dX\ \varphi(X)\chi(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)}\sigma(\theta-\theta',g), \end{aligned} \tag{3.80}$$

for some cut-off function $\chi \in C_c^\infty(\mathfrak{g})$ about $\mathrm{supp}_1(\check{\sigma}^1)$, i.e. $\chi \equiv 1$ on some relatively compact neighbourhood U of $\mathrm{supp}_1(\check{\sigma}^1)$ and $\chi \equiv 0$ on $\mathfrak{g}\setminus U'$ for some relatively compact neighbourhood $U' \supset U$. Clearly, both sides define holomorphic functions of θ, that are equal on $\mathfrak{g}^* \subset \mathfrak{g}^*_{\mathbb{C}}$, and thus are equal on $\mathfrak{g}^*_{\mathbb{C}}$. The holomorphicity of the last expression can be concluded from

$$\int_{\mathfrak{g}} dX\ \varphi(X)\chi(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)}\sigma(\theta-\theta',g) = \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)}\sigma(\theta-\theta',g)\mathscr{F}[\phi\chi](\theta') \tag{3.81}$$

and the observation that differentiation under the integral is permitted, which follows from $\sigma(\theta-(\,.\,),g)\mathscr{F}[\phi\chi] \in C^\infty_{\mathrm{PW}}(\mathfrak{g}^*)$. But, the last line in (3.80) is independent of χ due to the support properties of $\check{\sigma}^1$, which gives us

$$(H(\varphi)\sigma)(\theta,g) = \int_{\mathfrak{g}} dX\ \varphi(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)}\sigma(\theta-\theta',g). \tag{3.82}$$

Let us establish some important properties of the kernel cut-off operator.

Theorem III.25 (cp.[32], Theorem 3.7.):
The kernel cut-off operator $H : C_b^\infty(\mathfrak{g}) \times S^{K,m}_{\mathrm{PW},\rho,\delta} \to S^{K,m}_{\mathrm{PW},\rho,\delta}$ continuous. If $\rho > 0$ we have the asymptotic expansion

$$H(\varphi)\sigma \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{(-1)^{|\alpha|}}{\alpha!}((-i\partial_X)^\alpha \varphi)(0)\partial_\theta^\alpha \sigma \tag{3.83}$$

in $S^{K,m}_{\mathrm{PW},\rho,\delta}$.

__**Proof:**__
Since $S^{K,m}_{\mathrm{PW},\rho,\delta}$ is a Fréchet space it suffices to prove that the $||\ .\ ||_k^{(K,m,\rho,\delta)}$-seminorms of $H(\varphi)\sigma$ are bounded by the $||\ .|||_k^{(K,m,\rho,\delta)}$-seminorms of σ and the $||\ .\ ||_{\infty,k}$-seminorms of φ. By standard regularization techniques for oscillatory integrals, we have, for large enough $M \in \mathbb{N}_0$ and all $\alpha, \beta \in \mathbb{N}_0^n$:

$$\begin{aligned} &|(R^\alpha \partial_\theta^\beta H(\varphi)\sigma)(\theta, g)| \\ &= |(H(\varphi)(R^\alpha \partial_\theta^\beta \sigma))(\theta, g)| \\ &= \int_{\mathfrak{g}} \frac{dX}{\langle X\rangle^{2n}} \left((1-\Delta_X)^M \varphi\right)(X) \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)} \left((1-\Delta_{\theta'})^n \langle\theta'\rangle^{-2M} (R^\alpha \partial_\theta^\beta \sigma)(\theta-\theta', g)\right). \end{aligned} \tag{3.84}$$

The contribution of φ to the integral can be estimated by:

$$\begin{aligned} \left|\left((1-\Delta_X)^M \varphi\right)(X)\right| &\le \sum_{m=0}^M \binom{M}{m} |(\Delta_X^m \varphi)(X)| \\ &\le 2^M ||\varphi||_{\infty,2M}. \end{aligned} \tag{3.85}$$

Applying the Leibniz rule and the estimates

$$\begin{aligned} |\partial_{\theta'}^\gamma \langle\theta'\rangle^{-2M}| &\le C_{\gamma,M} \langle\theta'\rangle^{-2M-|\gamma|}, \\ |(R^\alpha \partial_\theta^{\beta+\gamma}\sigma)(\theta-\theta', g)| &\le C_{\alpha,\beta+\gamma} e^{H_K(\Im)(\theta-\theta')} \langle\theta-\theta'\rangle^{m-|\beta+\gamma|\rho+|\alpha|\delta} \\ &\underset{\substack{\theta' \in \mathfrak{g}^* \\ \text{Peetre's ineq.}}}{\le} C'_{\alpha,\beta+\gamma} e^{H_K(\Im)(\theta)} \langle\theta\rangle^{m-|\beta+\gamma|\rho+|\alpha|\delta} \langle\theta'\rangle^{|m-|\beta+\gamma|\rho+|\alpha|\delta|}, \end{aligned} \tag{3.86}$$

we get:

$$\begin{aligned} &\left|\left((1-\Delta_{\theta'})^n \langle\theta'\rangle^{-2M} (R^\alpha \partial_\theta^\beta \sigma)(\theta-\theta', g)\right)\right| \\ &\le C''_{\alpha,\beta,n,M} e^{H_K(\Im)(\theta)} \langle\theta\rangle^{m-|\beta|\rho+|\alpha|\delta} \langle\theta'\rangle^{-2M+|m|+(2n+|\beta|)\rho+|\alpha|\delta} ||\varphi||_{\infty,2M} ||\sigma||_{|\alpha|+|\beta|+2n}^{(K,m,\rho,\delta)}, \end{aligned} \tag{3.87}$$

and thus the seminorm estimate:

$$||H(\varphi)\sigma||_k^{(K,m,\rho,\delta)} \tag{3.88}$$

$$\leq C'''_{k,n,M} \underbrace{\left(\int_{\mathfrak{g}} \frac{dX}{\langle X\rangle^{2n}}\right)}_{=:C_{2n}<\infty} \underbrace{\left(\int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} \langle\theta'\rangle^{-2M+|m|+(2n+k)\rho+k\delta}\right)}_{<\infty \text{ for large } M} ||\varphi||_{\infty,2M} ||\sigma||^{(K,m,\rho,\delta)}_{k+2n}.$$

To obtain the asymptotic expansion, we consider the Taylor expansion of φ at $X=0$ of order $N-1$:

$$\varphi(X) = \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}(\partial^\alpha_X \varphi)(0) X^\alpha + \frac{1}{(N-1)!} \sum_{|\alpha|=N} \binom{N}{\alpha} X^\alpha \underbrace{\int_0^1 ds\ (1-s)^{N-1} (\partial^\alpha_X \varphi)(sX)}_{=:\varphi^\alpha_{(N)}(X)\in C^\infty_b(\mathfrak{g})}. \tag{3.89}$$

Plugging this expression into the kernel cut-off operator and integrating by parts, we find:

$$\begin{aligned}
(H(\varphi)\sigma)(\theta,g) &= \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}(\partial^\alpha_X\varphi)(0) \int_{\mathfrak{g}} dX \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)} \sigma(\theta-\theta',g) X^\alpha \\
&\quad + \frac{1}{(N-1)!} \sum_{|\alpha|=N} \binom{N}{\alpha} \int_{\mathfrak{g}} dX \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)} \sigma(\theta-\theta',g) X^\alpha \\
&\qquad\qquad \times \underbrace{\int_0^1 ds\ (1-s)^{N-1}(\partial^\alpha_X\varphi)(sX)}_{=:\varphi^\alpha_{(N)}(X)\in C^\infty_b(\mathfrak{g})} \\
&= \sum_{|\alpha|\leq N-1} \frac{(-1)^{|\alpha|}}{\alpha!}(-i\partial^\alpha_X\varphi)(0) \int_{\mathfrak{g}} dX \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)} (\partial^\alpha_\theta\sigma)(\theta-\theta',g) \\
&\quad + \frac{(-i)^N}{(N-1)!} \sum_{|\alpha|=N} \binom{N}{\alpha} \int_{\mathfrak{g}} dX \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{-i\theta'(X)} (\partial^\alpha_\theta\sigma)(\theta-\theta',g) \\
&\qquad\qquad \times \int_0^1 ds\ (1-s)^{N-1}(\partial^\alpha_X\varphi)(sX) \\
&= \sum_{|\alpha|\leq N-1} \frac{(-1)^{|\alpha|}}{\alpha!}(-i\partial^\alpha_X\varphi)(0) \underbrace{(\partial^\alpha_\theta\sigma)}_{\in S^{K,m-|\alpha|\rho}_{\mathrm{PW},\rho,\delta}}(\theta,g) \\
&\quad + \frac{(-i)^N}{(N-1)!} \sum_{|\alpha|=N} \binom{N}{\alpha} \underbrace{(H(\varphi^\alpha_{(N)})((\partial^\alpha_\theta\sigma)))}_{\in S^{K,m-N\rho}_{\mathrm{PW},\rho,\delta}}(\theta,g),
\end{aligned} \tag{3.90}$$

where the statement in the last line follows from lemma III.22 and the continuity property of H, which was shown before. The result follows by the definition of asymptotic expansions, i.e.: Given $\{m_k\}_{k=1}^\infty \subset \mathbb{R}$, s.t. $\lim_{k\to\infty} m_k = -\infty$ and $m := \max_{k\in\mathbb{N}} m_k$, and $\sigma_k \in S^{K,m_k}_{\mathrm{PW},\rho,\delta}$, $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}$, we say that $\sum_{k=1}^\infty a_k$ is asymptotic to a, $a \sim \sum_{k=1}^\infty a_k$, if

$$\forall M\in\mathbb{R} : \exists k_0\in\mathbb{N} : \forall k'\geq k_0 :\ a - \sum_{k=1}^{k'} a_k \in S^{K,M}_{\mathrm{PW},\rho,\delta}. \tag{3.91}$$

a is unique up to $S^{K,-\infty}_{\mathrm{PW},\rho,\delta}$ (smoothing symbols). $\square$

The preceding theorem implies the important

Corollary III.26 (cp.[32], Corollary 3.8.):
Given a cut-off function $\varphi \in C_c^\infty(\mathfrak{g})$ around $X = 0$, we have continuous operator

$$\mathrm{id} - H(\varphi) : S^{K,m}_{\mathrm{PW},\rho,\delta} \longrightarrow S^{K,-\infty}_{\mathrm{PW},\rho,\delta}. \tag{3.92}$$

Proof:
The Taylor expansion of $1-\varphi$ at $X = 0$ vanishes to infinite order, and $\mathrm{id} - H(\varphi) = H(1-\varphi)$. □

Now, we can state the main theorem of this section (cp.[32], Theorem 3.16.).

Theorem III.27 (Asymptotic completeness of the Paley-Wiener-Schwartz symbols):
The symbol spaces $S^{K,m}_{\mathrm{PW},\rho,\delta}$ are asymptotically complete in the following sense:

Given $\{m_k\}_{k=1}^\infty \subset \mathbb{R}$, s.t. $\lim_{k\to\infty} m_k = -\infty$ and $m := \max_{k\in\mathbb{N}} m_k$, and $\sigma_k \in C^\infty(G, S^{K,m_k}_{\mathrm{PW},\rho,\delta})$, there exists $\sigma \in C^\infty(G, S^{K,m}_{\mathrm{PW},\rho,\delta})$ s.t. $a \sim \sum_{k=1}^\infty a_k$. We call a a resummation *of $\sum_{k=1}^\infty a_k$.*

Before we prove the theorem, we need some lemmata, following the idea of proof for Volterra symbols by Krainer[32].

Lemma III.28:
Given $\beta \in \mathbb{N}_0^n$, $\varphi \in C_c^\infty(\mathfrak{g})$ and $\sigma \in S^{K,-(2(n+1)+|\beta|)}_{\mathrm{PW},\rho}(\mathfrak{g}^*_{\mathbb{C}})$, we have:

$$\sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}|(H((-i\partial_X)^\beta\varphi_c)(e^{ir\tau_n(\,\cdot\,)}\sigma))(\theta)| \leq k_n(\varphi,\beta)\frac{1}{c^{n+1}}||e^{ir\tau_n(\,\cdot\,)}\sigma||_0^{(2r,-(2(n+1)+|\beta|),\rho,\delta)}, \tag{3.93}$$

for some constant $k_n(\varphi,\beta) > 0$, where $c \in [0,\infty)$, $\varphi_c(X) = \varphi(cX)$, $r := \inf\{r' \in [0,\infty) \mid K \subset B_{r'}(0) = \{X \in \mathfrak{g} \mid |X| \leq r'\}\}$ and $e^{ir\tau_n(\,\cdot\,)}(\theta) := e^{ir\theta(\tau_n)}$ with τ_n the nth basis vector of $\mathfrak{g}$.

Proof:
First, observe that multiplication of σ with $e^{ir\tau_n(\,\cdot\,)}$ shifts the support of $\check{\sigma}$, $\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma] = \check{\sigma}(\,.\,+r\tau_n) =: \check{\sigma}_r$, s.t. $0 \in \mathfrak{g}$ is not an interior point of $\mathrm{supp}(\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma]) \subset K - r\tau_n \subset B_{2r}(0)$. Second, we have the equivalent estimates:

$$\sup_{\substack{\theta\in\mathfrak{g}^*_{\mathbb{C}}\\ |\alpha|=|\beta|+2(n+1)}} e^{-2r|\Im(\theta)|}|\theta^\alpha(e^{ir\tau_n(\,\cdot\,)}\sigma)(\theta)| < \infty \Leftrightarrow \sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{|\beta|+2(n+1)}|(e^{ir\tau_n(\,\cdot\,)}\sigma)(\theta)| < \infty. \tag{3.94}$$

Now, the assumptions imply that:

1. $\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma]_{|\,\mathfrak{g}_+} \equiv 0$, where $\mathfrak{g}_+ := \{X \in \mathfrak{g} \mid \theta_n(X) > 0\}$ (θ_n is dual to τ_n).
2. $\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma] \in C^{|\beta|+n+1}$, since

$$\begin{aligned}(\partial_X^\alpha\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma])(X) &= i^{|\alpha|}\int_{\mathfrak{g}^*}\frac{d\theta}{(2\pi)^n}\theta^\alpha e^{i\theta(X)}e^{ir\theta(\tau_n)}\sigma(\theta) \\ &= i^{|\alpha|}\int_{\mathfrak{g}^*}\frac{d\theta}{(2\pi)^n}\langle\theta\rangle^{-(n+1)}\left(\langle\theta\rangle^{n+1}\theta^\alpha e^{i\theta(X)}e^{ir\theta(\tau_n)}\sigma(\theta)\right)\end{aligned} \tag{3.95}$$

is finite for $|\alpha| \le \beta + n + 1$ by (3.94).

The Taylor expansion of $\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma]$ at $X = 0$ up to order $|\beta| + n$ reads:

$$\mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma](X) = \sum_{|\alpha|=|\beta|+n+1} R_\alpha(0)X^\alpha, \qquad |R_\alpha(0)| \le \frac{1}{\alpha!} \sup_{\substack{X\in K-r\tau_n\\|\gamma|=|\alpha|}} |\partial_X^\gamma \mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma](X)| \tag{3.96}$$

$$= \frac{1}{\alpha!} \sup_{\substack{X\in K\\|\gamma|=|\alpha|}} |(\partial_X^\gamma \check{\sigma})(X)|$$

since $\partial_X^\alpha \mathscr{F}^{-1}[e^{ir\tau_n(\,\cdot\,)}\sigma]$ vanishes at $X = 0$ for all $|\alpha| < |\beta| + n + 1$. Now, we come to the proof of (3.93):

$$\begin{aligned}
&\sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}|(H((-i\partial_X)^\beta\varphi_c)(e^{ir\tau_n(\,\cdot\,)}\sigma))(\theta)| \qquad (3.97)\\
&= \sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\Big|\int_{\mathfrak{g}} dX\ e^{-i\theta(X)}((-i\partial_X)^\beta\varphi_c)(X)\check{\sigma}_r(X)\Big|\\
&\le \sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\int_{\mathfrak{g}} dX\ e^{|\Im(\theta)||X|}|((-i\partial_X)^\beta\varphi_c)(X)\check{\sigma}_r(X)|\\
&\le \int_{\mathfrak{g}} dX\ |((-i\partial_X)^\beta\varphi_c)(X)\check{\sigma}_r(X)| = \int_{\mathfrak{g}} \frac{dX}{\langle X\rangle^{\frac{n+1}{2}}}\ \langle X\rangle^{\frac{n+1}{2}}|((-i\partial_X)^\beta\varphi_c)(X)\check{\sigma}_r(X)|\\
&\le \underbrace{\left(\int_{\mathfrak{g}} \frac{dX}{\langle X\rangle^{n+1}}\right)^{\frac{1}{2}}}_{=:C_{n+1}<\infty} \left(\int_{\mathfrak{g}} dX\ \langle X\rangle^{n+1}|((-i\partial_X)^\beta\varphi_c)(X)\check{\sigma}_r(X)|^2\right)^{\frac{1}{2}}\\
&\le C_{n+1}^{\frac{1}{2}}\left(\int_{\mathfrak{g}} dX\langle X\rangle^{n+1}|((-i\partial_X)^\beta\varphi_c)(X)|^2\Big(\sum_{|\alpha|=|\beta|+n+1}|X^\alpha||R_\alpha(0)|\Big)^2\right)^{\frac{1}{2}}\\
&\le C_{n+1}^{\frac{1}{2}} \sup_{\substack{X\in K\\|\gamma|=|\beta|+n+1}} |(\partial_X^\gamma\check{\sigma})(X)|\left(\int_{\mathfrak{g}} dX\langle X\rangle^{n+1}|((-i\partial_X)^\beta\varphi_c)(X)\sum_{|\alpha|=|\beta|+n+1}\frac{1}{\alpha!}X^\alpha|^2\right)^{\frac{1}{2}}\\
&\le C_{n+1}^{\frac{1}{2}}\frac{C_{n+1}}{(2\pi)^n}\left(\sup_{\theta\in\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{|\beta|+2(n+1)}|(e^{ir\tau_n(\,\cdot\,)}\sigma)(\theta)|\right)\\
&\qquad\qquad \times\left(\int_{\mathfrak{g}} dX\langle X\rangle^{n+1}|((-i\partial_X)^\beta\varphi_c)(X)\sum_{|\alpha|=|\beta|+n+1}\frac{1}{\alpha!}X^\alpha|^2\right)^{\frac{1}{2}}\\
&= C_{n+1}^{\frac{1}{2}}\frac{C_{n+1}}{(2\pi)^n}||e^{ir\tau_n(\,\cdot\,)}\sigma||_0^{(2r,-(2(n+1)+|\beta|),\rho,\delta)}\\
&\qquad\qquad \times\left(c^{-(3n+2)}\int_{\mathfrak{g}} dX\,\underbrace{\langle c^{-1}X\rangle^{n+1}}_{\le\langle X\rangle^{n+1}}|((-i\partial_X)^\beta\varphi)(X)\sum_{|\alpha|=|\beta|+n+1}\frac{1}{\alpha!}X^\alpha|^2\right)^{\frac{1}{2}}\\
&\le C_{n+1}^{\frac{1}{2}}\frac{C_{n+1}}{(2\pi)^n}||e^{ir\tau_n(\,\cdot\,)}\sigma||_0^{(2r,-(2(n+1)+|\beta|),\rho,\delta)}
\end{aligned}$$

$$\times \left(c^{-2(n+1)} \int_{\mathfrak{g}} dX \langle X\rangle^{n+1} |((-i\partial_X)^\beta \varphi)(X) \sum_{|\alpha|=|\beta|+n+1} \frac{1}{\alpha!} X^\alpha|^2 \right)^{\frac{1}{2}}$$

$$= \underbrace{C_{n+1}^{\frac{1}{2}} \frac{C_{n+1}}{(2\pi)^n} \left(\int_{\mathfrak{g}} dX \langle X\rangle^{n+1} |((-i\partial_X)^\beta \varphi)(X) \sum_{|\alpha|=|\beta|+n+1} \frac{1}{\alpha!} X^\alpha|^2 \right)^{\frac{1}{2}}}_{=:k_n(\varphi,\beta)}$$

$$\times\, c^{-(n+1)} ||e^{ir\tau_n(\ .\ )}\sigma||_0^{(2r,-(2(n+1)+|\beta|),\rho,\delta)} \qquad \square$$

Lemma III.29:

Let $N \in \mathbb{N}_0$, $\{M_{\alpha\beta}\}_{|\alpha|+|\beta|\leq N} \subset \mathbb{N}$, $\varphi \in C_c^\infty(\mathfrak{g})$ and $\sigma \in \mathscr{O}(\mathfrak{g}_{\mathbb{C}}^*, C^\infty(G))$, s.t.

$$\sup_{\substack{(g,\theta)\in G\times\mathfrak{g}_{\mathbb{C}}^* \\ |\alpha|+|\beta|\leq N}} e^{-r|\Im(\theta)|} \langle\theta\rangle^{2\left\lceil \frac{M_{\alpha\beta}}{2} \right\rceil + 2(n+1)} |(R^\alpha \partial_\theta^\beta \sigma)(\theta,g)| < \infty, \tag{3.98}$$

then we have:

$$\sup_{\substack{(g,\theta)\in G\times\mathfrak{g}_{\mathbb{C}}^* \\ |\alpha|+|\beta|\leq N}} e^{-2r|\Im(\theta)|} \langle\theta\rangle^{M_{\alpha\beta}} |(R^\alpha \partial_\theta^\beta H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma))(\theta,g)| \tag{3.99}$$

$$\leq\ \tilde{k}_n(\varphi, N, \{M_{\alpha\beta}\}) \frac{1}{c^{n+1}} \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}_{\mathbb{C}}^* \\ |\alpha|+|\beta|\leq N}} e^{-2r|\Im(\theta)|} \langle\theta\rangle^{2\left\lceil \frac{M_{\alpha\beta}}{2} \right\rceil + 2(n+1)} |(R^\alpha \partial_\theta^\beta (e^{ir\tau_n(\ .\ )}\sigma))(\theta,g)|$$

for some constant $\tilde{k}_n(\varphi, M, N) > 0$, where the notation of lemma III.28 is employed.

<u>Proof:</u>

The assumptions imply:

$$\sup_{\substack{(g,\theta)\in G\times\mathfrak{g}_{\mathbb{C}}^* \\ |\alpha|+|\beta|\leq N}} e^{-2r|\Im(\theta)|} \langle\theta\rangle^{2\left\lceil \frac{M_{\alpha\beta}}{2} \right\rceil + 2(n+1)} |(R^\alpha \partial_\theta^\beta (e^{ir\tau_n(\ .\ )}\sigma))(\theta,g)| < \infty, \tag{3.100}$$

as multiplication of σ with $e^{ir\tau_n(\ .\ )}$ shifts the support of $\check{\sigma}^1$, and modifies the decay properties of σ in the imaginary directions of $\mathfrak{g}_{\mathbb{C}}^*$ according to the Paley-Wiener-Schwartz theorem. Next, we observe, that the following properties hold, due to the definition of H:

1. $R^\alpha \partial_\theta^\beta H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma) = H(\varphi_c)(R^\alpha \partial_\theta^\beta (e^{ir\tau_n(\ .\ )}\sigma))$.

2. $\theta^\alpha (H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma))(\theta,g) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} H((-i\partial_X)^\beta \varphi_c)((\ .\ )^{\alpha-\beta} e^{ir\tau_n(\ .\ )}\sigma)(\theta,g)$.

3. $\langle\theta\rangle^{M_{\alpha\beta}} \leq \sum_{k=0}^{\left\lceil \frac{M_{\alpha\beta}}{2} \right\rceil} \binom{\left\lceil \frac{M_{\alpha\beta}}{2} \right\rceil}{k} \sum_{|\gamma|=k} \binom{k}{\gamma} \theta^{2\gamma}$.

Combining these properties with lemma III.28, we find:

$$\begin{aligned}
&\sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\|\alpha|+|\beta|\le N}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{M_{\alpha\beta}}|(R^\alpha\partial_\theta^\beta H(\varphi_c)(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)| \qquad (3.101)\\
&\le \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\|\alpha|+|\beta|\le N}} e^{-2r|\Im(\theta)|}\sum_{k=0}^{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}\binom{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}|\theta^{2\gamma}(R^\alpha\partial_\theta^\beta H(\varphi_c)(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)|\\
&\le \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\|\alpha|+|\beta|\le N}} e^{-2r|\Im(\theta)|}\sum_{k=0}^{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}\binom{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}\sum_{\zeta\le 2\gamma}\binom{2\gamma}{\zeta}\\
&\qquad\qquad\times|H((-i\partial_X)^\zeta\varphi_c)((\,.\,)^{2\gamma-\zeta}R^\alpha\partial_\theta^\beta(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)|\\
&\le \sup_{|\alpha|+|\beta|\le N} e^{-2r|\Im(\theta)|}\sum_{k=0}^{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}\binom{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}\sum_{\zeta\le 2\gamma}\binom{2\gamma}{\zeta}\\
&\qquad\qquad\times\sup_{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}}|H((-i\partial_X)^\zeta\varphi_c)((\,.\,)^{2\gamma-\zeta}R^\alpha\partial_\theta^\beta(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)|\\
&\underset{(3.93)}{\le} \sup_{|\alpha|+|\beta|\le N}\sum_{k=0}^{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}\binom{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}\sum_{\zeta\le 2\gamma}\binom{2\gamma}{\zeta}\frac{k(\varphi,\zeta)}{c^{n+1}}\\
&\qquad\qquad\times\sup_{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{|\zeta|+2(n+1)}|(\theta^{2\gamma-\zeta}R^\alpha\partial_\theta^\beta(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)|\\
&\le \sup_{|\alpha|+|\beta|\le N}\underbrace{\sum_{k=0}^{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}\binom{\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil}{k}\sum_{|\gamma|=k}\binom{k}{\gamma}\sum_{\zeta\le 2\gamma}\binom{2\gamma}{\zeta}\frac{k(\varphi,\zeta)}{c^{n+1}}}_{\le\frac{1}{c^{n+1}}\tilde{k}(\varphi,N,\{M_{\alpha\beta}\})}\\
&\qquad\qquad\times\sup_{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{2\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil+2(n+1)}|(R^\alpha\partial_\theta^\beta(e^{ir\tau_n(\,.\,)}\sigma))(\theta,g)|,
\end{aligned}$$

which concludes the proof. □

With the help of the preceding lemmata, we can prove a crucial convergence result for the symbol spaces (cp.[32], Proposition 3.14.).

Proposition III.30:

Given $\{m_k\}_{k=1}^\infty\subset\mathbb{R}$, *s.t.* $m_k\ge m_{k+1}\underset{k\to\infty}{\longrightarrow}-\infty$, *and a countable system of bounded sets* $\{S_{k_j}\}_{j\in\mathbb{N}}\subset$

$S^{K,m_k}_{\mathrm{PW},\rho,\delta}$ *for every* $k \in \mathbb{N}$, *then there exists a sequence* $\{c_i\}_{i=1}^\infty \subset [1,\infty)$, *with* $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$, *s.t.*

$$\sum_{i=k}^{\infty} \sup_{\sigma\in S_{i_j}} p(H(\varphi_{d_i})(e^{ir\tau_n(\ .\ )}\sigma)) < \infty, \tag{3.102}$$

for all $j,k \in \mathbb{N}$, *all continuous seminorms* p *on* $S^{B_{2r}(0),m_k}_{\mathrm{PW},\rho,\delta}$ *and all sequences* $\{d_i\}_{i=1}^\infty \subset [1,\infty)$ *with* $\forall i \in \mathbb{N} : d_i \geq c_i$. *Here, we use again the notation of lemma III.28.*

Proof:

Without loss of generality, we may assume that $\{m_k\}_{k=1}^\infty \subset \mathbb{R}_-$ and $S_{k_j} \subset S_{k_{j+1}}$ for all $j,k \in \mathbb{N}$. For all $l \in \mathbb{N}$, we define (ordered) seminorms

$$q_l^{2r,\rho,\delta}(\sigma) := \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\ |\alpha|+|\beta|\leq l}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{-m_l+|\beta|\rho-|\alpha|\delta}|(R^\alpha\partial_\theta^\beta\sigma)(\theta,g)|,\ q_l^{2r,\rho,\delta} \leq q_{l+1}^{2r,\rho,\delta}. \tag{3.103}$$

Using the preceding lemmata, we find for suitable σ and $c \in [1,\infty)$:

$$\begin{aligned} q_l^{2r,\rho,\delta}(H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma)) &= \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\ |\alpha|+|\beta|\leq l}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{-m_l+|\beta|\rho-|\alpha|\delta}|(R^\alpha\partial_\theta^\beta H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma))(\theta,g)| \\ &\leq \frac{1}{c^{n+1}}\tilde{k}(\varphi,l,\{M_{\alpha\beta}\}) \\ &\quad\times \sup_{\substack{(g,\theta)\in G\times\mathfrak{g}^*_{\mathbb{C}}\\ |\alpha|+|\beta|\leq l}} e^{-2r|\Im(\theta)|}\langle\theta\rangle^{2\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil+2(n+1)}|(R^\alpha\partial_\theta^\beta(e^{ir\tau_n(\ .\ )}\sigma))(\theta,g)|, \end{aligned} \tag{3.104}$$

where $M_{\alpha\beta} := \lceil -m_l + |\beta|\rho - |\alpha|\delta\rceil$. By assumption on the sequence $\{m_k\}_{k=1}^\infty$, we can find $i_0 \in \mathbb{N}$, $i_0 \geq l$, s.t. $2\left\lceil\frac{M_{\alpha\beta}}{2}\right\rceil+2(n+1) \leq -m_{i_0}+|\beta|\rho-|\alpha|\delta$, for $|\alpha|+|\beta| \leq l$, i.e. we need $m_{i_0}+2(n+2)+1 \leq m_l$. Thus, we get the estimate:

$$\forall l \in \mathbb{N} : \exists i_0 \in \mathbb{N} : \forall i \geq i_0 : q_l^{2r,\rho,\delta}(H(\varphi_c)(e^{ir\tau_n(\ .\ )}\sigma)) \leq \frac{\tilde{k}(\varphi,l,\{M_{\alpha\beta}\})}{c^{n+1}} q_i^{2r,\rho,\delta}(e^{ir\tau_n(\ .\ )}\sigma), \tag{3.105}$$

for $\sigma \in S^{K,m_i}_{\mathrm{PW},\rho,\delta}$. The existence of the sequence $\{c_i\}_{i=1}^\infty \subset [1,\infty)$ with the prescribed properties follows by induction. Following Krainer[32], we construct sequences $\{c_{l_i}\}_{i=1}^\infty \subset [1,\infty)$ for $l \in \mathbb{N}$, and take $\{c := c_{i_i}\}_{i=1}^\infty$:

1. Let $l = 1$: By (3.105), we can find a sequence $\{c_{1_i}\}_{i=1}^\infty \subset [1,\infty)$, $c_i < c_{i+1} \underset{i\to\infty}{\longrightarrow} \infty$, s.t. for all $i \in \mathbb{N}$ with $m_i + 2(n+2) + 1 \leq m_1$

$$\sup_{\sigma\in S_{i_1}} q_1^{2r,\rho,\delta}(H(\varphi_{d_i})(e^{ir\tau_n(\ .\ )}\sigma)) < 2^{-i} \tag{3.106}$$

holds for all $\{d_i\}_{i=1}^\infty \subset [1,\infty)$ with $\forall i \in \mathbb{N} : d_i \geq c_{1_i}$.

2. Let $\{c_{l_i}\}_{i=1}^\infty \subset [1,\infty)$ be constructed: By (3.105), we find a subsequence $\{c_{(l+1)_i}\}_{i=1}^\infty \subset$

$\{c_{l_i}\}_{i=1}^{\infty}$, s.t. for all $i \in \mathbb{N}$ with $m_i + 2(n+2) + 1 \leq m_{l+1}$

$$\sup_{\sigma \in S_{i_{l+1}}} q_{l+1}^{2r,\rho,\delta}(H(\varphi_{d_i})(e^{ir\tau_n(\ \cdot\ )}\sigma)) < 2^{-i} \tag{3.107}$$

holds for all $\{d_i\}_{i=1}^{\infty} \subset [1, \infty)$ with $\forall i \in \mathbb{N} : d_i \geq c_{(l+1)_i}$.

3. By construction, the diagonal sequence $\{c := c_{i_i}\}_{i=1}^{\infty} \subset [1, \infty)$, has the property $c_i \geq c_{l_i}$ for $i \geq l$ and $c_i < c_{i+1} \underset{i \to \infty}{\longrightarrow} \infty$.

4. Let $j, k \in \mathbb{N}$ and p be a continuous seminorm on $S_{\mathrm{PW},\rho,\delta}^{B_{2r}(0),m_k}$, then there exist l_0 s.t. the restriction of p to $S_{\mathrm{PW},\rho,\delta}^{B_{2r}(0),m_i}$ is dominated by $q_{l_0}^{2r,\rho,\delta}$ with a constant independent of i and $S_{i_j} \subset S_{i_{l_0}}$ for almost all $i \in \mathbb{N}$. These assertions follow from the inclusion properties of the symbol spaces (see (3.66)). Employing the continuity of H (see theorem III.25), we conclude that the series (3.102) is indeed convergent for given data $j, k \in \mathbb{N}$ and p. □

Proof (of Theorem III.27):

Without loss of generality, we may assume $m_k \geq m_{k+1} \underset{k \to \infty}{\longrightarrow} -\infty$. For $j, k \in \mathbb{N}$, we define $S_{k_j} := \{(R_1^{\alpha}\sigma_k)(g) \mid g \in G,\ |\alpha| \leq j\} \subset S_{\mathrm{PW},\rho,\delta}^{K,m_k}$. Here, R_1 is the right differential in the first group variable. Since $\sigma_k \in C^{\infty}(G, S_{\mathrm{PW},\rho,\delta}^{K,m_k})$, we know that the sets S_{k_j} are bounded in $S_{\mathrm{PW},\rho,\delta}^{K,m_k}$. Now, we choose a cut-off function $\varphi \in C_c^{\infty}(\mathfrak{g})$ around $X = 0$, and apply proposition III.30 to obtain a sequence $\{c_i\}_{i=1}^{\infty} \subset [1, \infty)$, s.t.

$$\sum_{i=k}^{\infty} \sup \left\{ p\left((H(\varphi_{c_i})(e^{ir\tau_n(\ \cdot\ )} R^{\alpha}\sigma))(g) \right) \ \middle|\ g \in G,\ |\alpha| \leq j \right\} < \infty \tag{3.108}$$

for all continuous seminorms p on $S_{\mathrm{PW},\rho,\delta}^{B_{2r}(0),m_k}$. Therefore, the sum

$$a^{(r)} := \sum_{i=1}^{\infty} H(\varphi_{c_i})(e^{ir\tau_n(\ \cdot\ )}\sigma) \tag{3.109}$$

is unconditionally convergent in $C^{\infty}(G, S_{\mathrm{PW},\rho,\delta}^{K-r\tau_n,m})$. Now, we define:

$$a := e^{-ir\tau_n(\ \cdot\ )} a^{(r)}, \tag{3.110}$$

which is in $C^{\infty}(G, S_{\mathrm{PW},\rho,\delta}^{K,m})$. It follows that the $a \sim \sum_{k=1}^{\infty} a_k$:

$$\begin{aligned} a - \sum_{i=1}^{k} a_i &= e^{-ir\tau_n(\ \cdot\ )} \left(a^r - \sum_{i=1}^{k} e^{ir\tau_n(\ \cdot\ )} a_i \right) \\ &= e^{-ir\tau_n(\ \cdot\ )} \sum_{i=k+1}^{\infty} H(\varphi_{c_i})(e^{ir\tau_n(\ \cdot\ )}\sigma) + e^{-ir\tau_n(\ \cdot\ )} \sum_{i=1}^{k} (\mathrm{id} - H(\varphi_{c_i}))(e^{ir\tau_n(\ \cdot\ )}\sigma) \end{aligned} \tag{3.111}$$

$$= \underbrace{e^{-ir\tau_n(\,\cdot\,)} \sum_{i=k+1}^{\infty} H(\varphi_{c_i})(e^{ir\tau_n(\,\cdot\,)}\sigma)}_{\in C^\infty\left(G, S^{K,m_{k+1}}_{\mathrm{PW},\rho,\delta}\right)} + \underbrace{e^{-ir\tau_n(\,\cdot\,)} \sum_{i=1}^{k} H(1-\varphi_{c_i})(e^{ir\tau_n(\,\cdot\,)}\sigma)}_{\in C^\infty\left(G, S^{K,-\infty}_{\mathrm{PW},\rho,\delta}\right) \text{ Cor. III.26}}. \qquad \square$$

We conclude the subsection by showing that the operator product of two Weyl quantisations $F^{W,\varepsilon}_\sigma,\ F^{W,\varepsilon}_\tau$ of Paley-Wiener-Schwartz symbols $\sigma \in S^{K,m}_{\mathrm{PW},\rho,\delta}, \tau \in S^{K',m'}_{\mathrm{PW},\rho,\delta}$, assuming that $\varepsilon \in (0,1]$ is small enough, has a formal expansion in ε that qualifies as an asymptotic series for certain values of $0 \le \delta \le \rho \le 1$. Moreover the series has finitely many non-vanishing terms if σ and τ are polynomial. Unfortunately, we can (so far) not establish that this series is asymptotic to the dequantisation of the operator product, because the image of the Weyl quantisation is not obviously closed under products[47]. To this end, we recall that the group product in $V \subset G$ can be pulled back to $U \subset \mathfrak{g}$, and the Dynkin-Baker-Campbell-Hausdorff formula tells us, that we may write

$$\begin{aligned} X_{hg} = \exp^{-1}(hg) = \exp^{-1}(X_h) * \exp^{-1}(X_g) &= X_h + X_g + \sum_{k=1}^{\infty} P_k(X_h, X_g) \\ &= X_h + X_g + \frac{1}{2}[X_h, X_g] + \frac{1}{12}([X_h,[X_h,X_g]] + [X_g,[X_g,X_h]]) + \text{higher orders}, \end{aligned} \tag{3.112}$$

for sufficiently small $X_h, X_g \in U$ (cf.[48] for convergence properties of (3.112)). Here, the $P_k,\ k \in \mathbb{N}$, are Lie-Polynomials. The ε-scaled version of the product, $\exp(\varepsilon(X_h *_\varepsilon X_g)) = \exp(\varepsilon X_h)\exp(\varepsilon X_g)$, is:

$$\begin{aligned} \varepsilon^{-1}((\varepsilon X_h) * (\varepsilon X_g)) = X_h *_\varepsilon X_\varepsilon &= X_h + X_g + \sum_{k=1}^{\infty} \varepsilon^k P_k(X_h, X_g) \\ &= X_h + X_g + \frac{\varepsilon}{2}[X_h, X_g] + \frac{\varepsilon^2}{12}([X_h,[X_h,X_g]] + [X_g,[X_g,X_h]]) + O(\varepsilon^3). \end{aligned} \tag{3.113}$$

If we apply this formula to the twisted convolution (2.5) of the Weyl quantisations of $F^{W,\varepsilon}_\sigma,\ F^{W,\varepsilon}_\tau$, we get for ε small enough:

$$\begin{aligned} &\left(F^{W,\varepsilon}_\sigma * F^{W,\varepsilon}_\tau\right)(h,g) \\ &= \int_G dh'\ F^{W,\varepsilon}_\sigma(h',g) F^{W,\varepsilon}_\tau(h'^{-1}h, h'^{-1}g) \\ &= \int_G dh'\ F^\varepsilon_\sigma(h', \sqrt{h'^{-1}}g) F^\varepsilon_\tau(h'^{-1}h, \sqrt{h^{-1}h'}h'^{-1}g) \\ &= \varepsilon^{-2n} \int_G dh'\ \check{\sigma}^1\big(\varepsilon^{-1}X_{h'}, \exp(-\tfrac{1}{2}X_{h'})g\big) \check{\tau}^1\big(\varepsilon^{-1}X_{h'^{-1}h}, \exp(-\tfrac{1}{2}X_{h'^{-1}h})\exp(-X_{h'})g\big) \\ &= \varepsilon^{-2n} \int_G dh'\ \check{\sigma}^1\big(\varepsilon^{-1}X_{h'}, \exp(-\tfrac{1}{2}X_{h'})g\big) \\ &\qquad \times \check{\tau}^1\big(\varepsilon^{-1}((-X_{h'}) * X_h), \exp(-\tfrac{1}{2}((-X_{h'}) * X_h))\exp(-X_{h'})g\big) \\ &= \varepsilon^{-2n} \int_{\mathfrak{g}} dX_{h'}\ j(X_{h'})^2\ \check{\sigma}^1\big(\varepsilon^{-1}X_{h'}, \exp(-\tfrac{1}{2}X_{h'})g\big) \\ &\qquad \times \check{\tau}^1\big(\varepsilon^{-1}((-X_{h'}) * X_h), \exp(-\tfrac{1}{2}((-X_{h'}) * X_h))\exp(-X_{h'})g\big) \end{aligned} \tag{3.114}$$

$$
\begin{aligned}
&= \varepsilon^{-n} \int_{\mathfrak{g}} dX_{h'}\ j(\varepsilon X_{h'})^2\ \check{\sigma}^1\big(X_{h'}, \exp(-\tfrac{\varepsilon}{2} X_{h'}) g\big) \\
&\qquad \times \check{\tau}^1\big(\varepsilon^{-1}((-\varepsilon X_{h'}) * X_h), \exp(-\tfrac{1}{2}((-\varepsilon X_{h'}) * X_h)) \exp(-\varepsilon X_{h'}) g\big) \\
&= \varepsilon^{-n} \int_{\mathfrak{g}} dX_{h'}\ j(\varepsilon X_{h'})^2\ \check{\sigma}^1\big(X_{h'}, \exp(-\tfrac{\varepsilon}{2} X_{h'}) g\big) \\
&\qquad \times \check{\tau}^1\big(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)), \exp(-\tfrac{\varepsilon}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h))) \exp(-\varepsilon X_{h'}) g\big) \\
&= \varepsilon^{-n} \int_{\mathfrak{g}} dX_{h'}\ j(\varepsilon X_{h'})^2\ \check{\sigma}^1\big(X_{h'}, \exp(-\tfrac{\varepsilon}{2} X_{h'}) \exp(\tfrac{\varepsilon}{2}(\varepsilon^{-1} X_h)) \exp(-\tfrac{1}{2}(X_h)) g\big) \\
&\qquad \times \check{\tau}^1\big(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)), \\
&\qquad\qquad \exp(-\tfrac{\varepsilon}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h))) \exp(-\varepsilon X_{h'}) \exp(\tfrac{\varepsilon}{2}(\varepsilon^{-1} X_h)) \exp(-\tfrac{1}{2}(X_h)) g\big) \\
&= \varepsilon^{-n} \int_{\mathfrak{g}} dX_{h'}\ j(\varepsilon X_{h'})^2\ \check{\sigma}^1\big(X_{h'}, \exp(\varepsilon((-\tfrac{1}{2} X_{h'}) *_\varepsilon (\tfrac{1}{2}(\varepsilon^{-1} X_h)))) \exp(-\tfrac{1}{2}(X_h)) g\big) \\
&\qquad \times \check{\tau}^1\big(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)), \\
&\qquad\qquad \exp(\varepsilon((-\tfrac{1}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h))) *_\varepsilon (((-X_{h'}) *_\varepsilon (\tfrac{1}{2}(\varepsilon^{-1} X_h)))))) \exp(-\tfrac{1}{2}(X_h)) g\big),
\end{aligned}
$$

where we switched integration from G to $\mathfrak{g}$ by means of the exponential map (see (3.45)), which is permitted due to the support properties of $\check{\sigma}^1$ and $\check{\tau}^1$, changed integration variables $X_{h'} \mapsto \varepsilon^{-1} X_{h'}$, and successively replaced the group product in V by its ε-scaled pullback $*_\epsilon$ in U.

Now, we would like to write $\left(F_\sigma^{W,\varepsilon} * F_\tau^{W,\varepsilon}\right)(h,g) = F_\rho^{W,\epsilon}(h,g)$ for some $\rho \in S^{K+K',m+m'}_{\mathrm{PW},\rho,\delta}$, which would define the twisted product of symbols $\rho = \sigma \star_\varepsilon \tau$. The shift in the support $(K,K') \mapsto K+K'$ is to be expected, because of the relation of $\star_\epsilon$ to the convolution of $\check{\sigma}^1$ and $\check{\tau}^1$. Unfortunately, as already mentioned above, the last line in (3.114), is not obviously of the form $F_\rho^{W,\epsilon}(h,g)$. Nevertheless, if we apply (3.113), we can deduce an asymptotic series in $S^{K+K',m+m'}_{\mathrm{PW},\rho,\delta}$ by successive integration by parts. To this end, we explicitly write out the inverse Fourier transforms in (3.114):

$$
\begin{aligned}
&\left(F_\sigma^{W,\varepsilon} * F_\tau^{W,\varepsilon}\right)(h,g) \qquad (3.115) \\
&= \varepsilon^{-n} \int_{\mathfrak{g}} dX_{h'}\ j(\varepsilon X_{h'})^2 \int_{\mathfrak{g}^*} \frac{d\theta}{(2\pi)^n} e^{i\theta(X_{h'})} \sigma(\theta, \exp(\varepsilon((-\tfrac{1}{2} X_{h'}) *_\varepsilon (\tfrac{1}{2}(\varepsilon^{-1} X_h)))) \exp(-\tfrac{1}{2}(X_h)) g) \\
&\quad \times \int_{\mathfrak{g}^*} \frac{d\theta'}{(2\pi)^n} e^{i\theta'(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)))} \\
&\qquad \times \tau(\theta', \exp(\varepsilon((-\tfrac{1}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h))) *_\varepsilon (((-X_{h'}) *_\varepsilon (\tfrac{1}{2}(\varepsilon^{-1} X_h)))))) \exp(-\tfrac{1}{2}(X_h)) g).
\end{aligned}
$$

From the Dynkin-Baker-Campbell-Hausdorff formula (3.112), and the fact that the $P_k,\ k \in \mathbb{N}$, are Lie-Polynomials, we infer that

$$
(-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h) = -(X_{h'} - \varepsilon^{-1} X_h) + \sum_{k=1}^{\infty} \varepsilon^k P'_k(X_{h'}, X_{h'} - \varepsilon^{-1} X_h), \qquad (3.116)
$$

for some (new) Lie-Polynomials $P'_k,\ k \in \mathbb{N}$.

Therefore, we can achieve the following rewriting of $e^{i\theta'(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)))}$:

$$
\begin{aligned}
e^{i\theta'(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)))} &= e^{i\theta'(((-X_{h'}) *_\varepsilon (\varepsilon^{-1} X_h)) + (X_{h'} - \varepsilon^{-1} X_h))} e^{-i\theta'(X_{h'} - \varepsilon^{-1} X_h)} \qquad (3.117) \\
&= \prod_{k=1}^{\infty} e^{i\varepsilon^k \theta'(P'_k(X_{h'}, X_{h'} - \varepsilon^{-1} X_h))} e^{-i\theta'(X_{h'} - \varepsilon^{-1} X_h)}
\end{aligned}
$$

$$= \prod_{k=1}^{\infty} \varepsilon^k Q_k(\varepsilon, X_{h'}, \theta', \partial_{\theta'}) e^{-i\theta'(X_{h'}-\varepsilon^{-1}X_h)},$$

where $Q_k(\varepsilon, X_{h'}, \theta', \partial_{\theta'})$, $k \in \mathbb{N}$, are differential operators of infinite order. If we continue by performing Taylor expansions of $j(\varepsilon X_{h'})^2$, $\sigma(\theta, \exp(\varepsilon((-\frac{1}{2}X_{h'}) *_\varepsilon (\frac{1}{2}(\varepsilon^{-1}X_h)))) \exp(-\frac{1}{2}(X_h))g)$ and $\tau(\theta', \exp(\varepsilon((-\frac{1}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1}X_h))) *_\varepsilon (((-X_{h'}) *_\varepsilon (\frac{1}{2}(\varepsilon^{-1}X_h)))))) \exp(-\frac{1}{2}(X_h))g)$ in ε, trade all $X_{h'}$- and $(X_{h'} - \varepsilon^{-1}X_h)$-dependence by differentiation of $e^{i\theta(X_{h'})}$ and $e^{-i\theta'(X_{h'}-\varepsilon^{-1}X_h)}$ for θ- and θ'-differentials, and finally perform repeated integrations by parts, culminating in an expression where the $dX_{h'}$ integration gives an oscillatory integral representation of $\delta^{(n)}(\theta - \theta')$, we will get an infinite sum in orders of ε

$$\left(F_\sigma^{W,\varepsilon} *_{\mathrm{L}} F_\tau^{W,\varepsilon}\right) \overset{?}{\sim} \sum_{k=0}^{\infty} \varepsilon^k C_k^{W,\varepsilon}(\sigma, \tau) \tag{3.118}$$

consisting of expressions of the form $C_k^{W,\varepsilon}(\sigma, \tau)$, where each order ε^k can be written as products of differentials of σ and τ of total order $\leq k$ each. Due to the fact, that $j \neq 1$ for general compact Lie groups G, the differential operations on σ and τ will not solely be determined by the Dynkin-Baker-Campbell-Hausdorff formula, and thus by the Poisson bracket (3.60), but also be affected by differential operators determined by the structure of the (positive) roots of G. Nevertheless, it follows from the Taylor series of j that those additional differential operators will lead to lower order terms in $S^{K+K',m+m'}_{\mathrm{PW},\rho,\delta}$, than those coming from the Poisson bracket, if $\rho > \delta$. Taking a closer look at the formula for the Poisson bracket (3.60) and (3.76), we realise that the condition $\rho > \delta$, known from the $\mathbb{R}^n$-case, has to be supplemented by $\rho > \frac{1}{2}$ for the expression (3.118) to qualify as an asymptotic sum. We note that, by theorem III.27, there exists a resummation of the right hand side of (3.118), which is unique up to smoothing symbols in $S^{K+K',-\infty}_{\mathrm{PW},\rho,\delta}$. Thus, from a practical point of view it would suffice to show that the difference between the operators defined by the left hand side and the resummation of the right hand side is sufficiently "small" in a sense to be made precise, if asymptoticity in $S^{K+K',m+m'}_{\mathrm{PW},\rho,\delta}$ were to fail.

Finally, let us be a bit more explicit, and give the expression for (3.118) up to order ε. Additionally, we provide the expression up to order ε^2 before partial integration:

$$\begin{aligned} &j(\varepsilon X_{h'})^2 e^{i\theta'(((-X_{h'})*_\varepsilon(\varepsilon^{-1}X_h)))} \sigma(\theta, \exp(\varepsilon((-\tfrac{1}{2}X_{h'})*_\varepsilon(\tfrac{1}{2}(\varepsilon^{-1}X_h)))) \exp(-\tfrac{1}{2}(X_h))g) \\ &\quad\times \tau(\theta', \exp(\varepsilon((-\tfrac{1}{2}((-X_{h'}) *_\varepsilon (\varepsilon^{-1}X_h))) *_\varepsilon (((-X_{h'}) *_\varepsilon (\tfrac{1}{2}(\varepsilon^{-1}X_h)))))) \exp(-\tfrac{1}{2}(X_h))g) \end{aligned} \tag{3.119}$$

$$\begin{aligned} \sim_{O(\varepsilon^3)} \quad & e^{-i\theta'(X_{h'}-\varepsilon^{-1}X_h)} \Bigg(\sigma(\theta, \sqrt{h^{-1}}g)\tau(\theta', \sqrt{h^{-1}}g) \\ &+ \frac{\varepsilon}{2}\Bigg(i\theta'([X_{h'}, X_{h'} - \varepsilon^{-1}X_h])\sigma(\theta, \sqrt{h^{-1}}g)\tau(\theta', \sqrt{h^{-1}}g) \\ &- (R_{X_{h'}-\varepsilon^{-1}X_h}\sigma)(\theta, \sqrt{h^{-1}}g)\tau(\theta', \sqrt{h^{-1}}g) - \sigma(\theta, \sqrt{h^{-1}}g)(R_{X_{h'}}\tau)(\theta', \sqrt{h^{-1}}g)\Bigg) \\ &+ \frac{\varepsilon^2}{4}\Bigg(\Bigg(-\frac{4}{3}\sum_{\alpha \in R^+} \alpha(X_{h'})^2 - \frac{1}{2}(\theta'([X_{h'}, X_{h'} - \varepsilon^{-1}X_h]))^2 \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{3} i\theta'(2[X_{h'}, X_{h'} - \varepsilon^{-1}X_h]_{(2)} + [X_{h'} - \varepsilon^{-1}X_h, X_{h'}]_{(2)})\Big)\sigma(\theta, \sqrt{h^{-1}}g)\tau(\theta', \sqrt{h^{-1}}g) \\
&+ \frac{1}{2}\Big(\sigma(\theta, \sqrt{h^{-1}}g)\big((R^2_{X_{h'}}\tau)(\theta', \sqrt{h^{-1}}g) + (R_{[X_{h'}, X_{h'}-\varepsilon^{-1}X_h]}\tau)(\theta', \sqrt{h^{-1}}g)\big) \\
&+ \big((R^2_{X_{h'}-\varepsilon^{-1}X_h}\sigma)(\theta, \sqrt{h^{-1}}g) + (R_{[X_{h'}, X_{h'}-\varepsilon^{-1}X_h]}\sigma)(\theta, \sqrt{h^{-1}}g)\big)\tau(\theta', \sqrt{h^{-1}}g)\Big) \\
&+ (R_{X_{h'}-\varepsilon^{-1}X_h}\sigma)(\theta, \sqrt{h^{-1}}g)(R_{X_{h'}}\tau)(\theta', \sqrt{h^{-1}}g) \\
&- i\theta'([X_{h'}, X_{h'} - \varepsilon^{-1}X_h]) \\
&\quad \times \Big((R_{X_{h'}-\varepsilon^{-1}X_h}\sigma)(\theta, \sqrt{h^{-1}}g)\tau(\theta', \sqrt{h^{-1}}g) + \sigma(\theta, \sqrt{h^{-1}}g)(R_{X_{h'}}\tau)(\theta', \sqrt{h^{-1}}g)\Big)\Big)\Big).
\end{aligned}
$$

Integrating this expression at order ε, we have:

$$
\begin{aligned}
\left(F^{W,\varepsilon}_{\sigma} * F^{W,\varepsilon}_{\tau}\right) \quad &\sim_{O(\varepsilon^2)} \ F^{W,\varepsilon}_{\sigma\tau} - \frac{i\varepsilon}{2} F^{W,\varepsilon}_{\{\sigma,\tau\}_{T^*G}}, \\
\sigma \star_\varepsilon \tau \quad &\sim_{O(\varepsilon^2)} \ \sigma\tau - \frac{i\varepsilon}{2}\{\sigma, \tau\}_{T^*G}
\end{aligned}
\tag{3.120}
$$

as expected from theorem III.14.

B. Scaled Fourier transforms, the Stratonovich-Weyl transform and coadjoint orbits

The global and local definitions of pseudo-differential operators we introduced in the previous subsection appear to be intimately tied to the natural representation $L^2(G)$ of the transformation group C^*-algebra $C(G) \rtimes_{\alpha_{\mathrm{L}}} G$, and its regularity properties w.r.t. the group translations $U_g,\ g \in G$, up to this point. Especially, the expansion (3.118) associates the ε-dependence with the "momentum variables" $P_X,\ X \in \mathfrak{g}$, in (3.1). But, in view of applications in Born-Oppenheimer reduction schemes and adiabatic perturbation theory it seems to be useful to be able to shift the ε-dependence to the variables dual to the "momenta", thereby changing from the microscopic scale to the macroscopic scale ($X'_g = \varepsilon X_g,\ g' = \exp(\varepsilon X_g)$). Furthermore, it can be advantageous to switch the roles of "momenta" and "positions" by means of a suitable integral transform on $L^2(G)$, as seen form[13] (Section 5).

To exemplify this point, we recall how the Fourier transform, and its ε-scaled version, affect the case of pseudo-differential operators on $\mathbb{R}^n$. Namely, if we consider a symbol $\sigma \in S^m_{\rho,\delta}$ or $\mathscr{S}'(\mathbb{R}^{2n})$, and its action on $\mathscr{S}(\mathbb{R}^n)$ via Weyl quantisation,

$$
(A_\sigma\Psi)(q) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^{2n}} dx\ d\xi\ \sigma(\tfrac{1}{2}(q+x), \xi)e^{\frac{i}{\varepsilon}\xi\cdot(q-x)}\Psi(x),
\tag{3.121}
$$

which is adapted to the representation of the commutation relations (3.2) by $Q = q\cdot$ and $P = -i\varepsilon\nabla_q$, we may interchanges the roles of Q and P w.r.t. to the Weyl quantisation by applying the ε-scaled Fourier transform,

$$
\mathscr{F}_\varepsilon[\Psi](p) = \hat{\Psi}_{(\varepsilon)}(p) := \int_{\mathbb{R}^n} dq\ e^{-\frac{i}{\varepsilon}p\cdot q}\Psi(q),
\tag{3.122}
$$

$$\mathscr{F}_\varepsilon^{-1}[\Phi](q) = \check{\Phi}^{(\varepsilon)}(q) := \int_{\mathbb{R}^n} \frac{dp}{(2\pi\varepsilon)^n}\; e^{\frac{i}{\varepsilon}p\cdot q}\Phi(p),$$

to obtain

$$(\hat{A}_\sigma \hat{\Psi}_{(\varepsilon)})(p) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^{2n}} dx\; d\xi\; \sigma(x, \tfrac{1}{2}(p+\xi))e^{-\frac{i}{\varepsilon}(p-\xi)\cdot x}\hat{\Psi}_{(\varepsilon)}(\xi), \tag{3.123}$$

which is adapted to the representation of (3.2) by $Q = -i\varepsilon\nabla_p$ and $P = p\cdot$. Consider, for example, a standard Born-Oppenheimer type Hamiltonian

$$H = -\frac{\varepsilon^2}{2}(\nabla_q + iA(q))^2 \otimes \mathbb{1}_{\mathfrak{H}_f} + V(q) \tag{3.124}$$

acting on the coupled quantum system $\mathfrak{H} = L^2(\mathbb{R}^n) \otimes \mathfrak{H}_f$. (3.124) is obtained as Weyl quantisation (3.121) of the ε-dependent symbol

$$\sigma_H(q,p) = \frac{1}{2}(p - \varepsilon A(q))^2 \otimes \mathbb{1}_{\mathfrak{H}_f} + V(q). \tag{3.125}$$

On the Fourier transformed side, the Hamiltonian takes the form (3.123):

$$\begin{aligned} \hat{H} &= \frac{\varepsilon^2}{2}(p - A(i\nabla_p))^2 \otimes \mathbb{1}_{\mathfrak{H}_f} + V(i\nabla_p) \quad \text{(unscaled)}, \\ \hat{H}_\varepsilon &= \frac{1}{2}(p - \varepsilon A(i\varepsilon\nabla_p))^2 \otimes \mathbb{1}_{\mathfrak{H}_f} + V(i\varepsilon\nabla_p) \quad (\varepsilon\text{-scaled}). \end{aligned} \tag{3.126}$$

While these observation are almost trivial at this level, equation (3.124) and (3.126) illustrate the fact, that it is beneficial to have suitable integral transforms, possibly with ε-scaling, at hand to decide whether a given operator can be written as Weyl quantisation of a, possibly operator valued, symbol to apply Born-Oppenheimer reduction or space adiabatic perturbation theory. A less trivial example is given by a Hamiltonian with periodic potential V_Γ ($\Gamma \subset \mathbb{R}^n$ is the periodicity lattice of V_Γ) and slowly varying external electromagnetic fields A, ϕ considered by Panati, Teufel and Spohn[13,49]:

$$H = \frac{1}{2}(-i\nabla_q - A(\varepsilon q))^2 + V_\Gamma(q) + \phi(\varepsilon q), \tag{3.127}$$

which can be rewritten as Weyl quantisation of an operator valued symbol by means of the Bloch-Floquet transform $U : L^2(\mathbb{R}^n) \to \mathfrak{H}_{\Gamma^*}$ (cf.[11], Chapter 1.10):

$$U[\Psi](p,q) := \sum_{\gamma\in\Gamma} e^{-ip\cdot(q+\gamma)}\Psi(q+\gamma),\;\; (q,p) \in \mathbb{R}^n,\;\; \Psi \in \mathscr{S}(\mathbb{R}^n), \tag{3.128}$$

where $\mathfrak{H}_{\Gamma^*} = \{\Phi \in L^2_{\text{loc}}(\mathbb{R}^n, L^2(\mathbb{T}^n)) \mid \Phi(p+\gamma^*, q) = e^{-i\gamma^*\cdot q}\Phi(p,q)\}$ and Γ^* is the lattice dual to Γ. Applying U to (3.127) gives:

$$UHU^* = \frac{1}{2}(-i\nabla_q^{\text{per}} + p + A(i\varepsilon\nabla_p))^2 + V_\Gamma(q) + \phi(i\varepsilon\nabla_p), \tag{3.129}$$

which can be understood as the Weyl quantisation of a Γ^*-equivariant symbol with values in bounded operators from $H^2(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$, and thus makes (3.127) accessible to space adiabatic perturbation theory.

This said, we return to the case of compact and simply connected Lie groups[50], where we apply the Stratonovich-Weyl-Fourier transform introduced by Figueroa, Gracia-Bondía and Várilly[8,34], based on ideas of Stratonovich[51], to the pseudo-differential operators defined in the previous section, which gives rise to an alternative to the common Fourier transform from $L^2(G)$ to $L^2(\hat{G})$, and additionally makes the effect of (radial) scaling particularly transparent. Since this transform is somewhat non-standard, we recall the main steps of its construction in some detail:

According to[8], we construct the Stratonovich-Weyl operator $\Delta^\pi : \mathscr{O}_\pi \to \mathrm{End}(V_\pi)$, which allows us to map functions on $\mathscr{O}_\pi$ to operator in V_π, via the coherent state formalism for compact semisimple Lie groups[7,52,53]:

1. For a unitary irreducible representation, $\pi \in \hat{G}$, we choose the corresponding (real) highest weight $\lambda_\pi \in \mathfrak{t}^* \subset \mathfrak{g}^*$. Let $\mathscr{O}_\pi = \{\theta \in \mathfrak{g}^* \mid \exists g \in G : \theta = Ad_g^*(\lambda_\pi)\} \subset \mathfrak{g}^*$ be the coadjoint orbit of G through λ_π. $\mathscr{O}_\pi \cong G/G_{\lambda_\pi}$ is homogeneous space, where G_{λ_π} is the stabiliser of λ_π. It is the content of the Borel-Weil theorem that the correspondence $\pi \leftrightarrow \lambda_\pi$ is one-to-one up to unitary equivalence[54].

2. Next, we choose a normalised weight vector $v_\pi \in V_\pi$ of λ_π. Moreover, we define the equivariant momentum map
$$J_\pi(v)(X) := \frac{1}{2\pi i}(v, d\pi(X)v)_{V_\pi},\ v \in V_\pi, X \in \mathfrak{g}, \tag{3.130}$$
which satisfies $J_\pi(\pi(g)v) = Ad_g^*(J_\pi(v)),\ g \in G,,\ J_\pi(v_\pi) = \lambda_\pi$ and $J_\pi^{-1}(\{\lambda_\pi\}) = \{zv_\pi \mid z \in \mathbb{C}, |z| = 1\}$ because the weight space of λ_π is one-dimensional. Then, we have $J_\pi^{-1}(\mathscr{O}_\pi) = \{\pi(g)v_\pi \mid g \in G\}$.

3. For $\theta \in \mathscr{O}_\pi$ we choose $g_\theta \in G$ s.t. $Ad_{g_\theta}^*(\lambda_\pi) = \theta$ and $g_{\lambda_\pi} = e$. $\theta \mapsto g_\theta$ is a measurable section w.r.t. to the Liouville measure $d\mu_\pi$ on $\mathscr{O}_\pi$ ($\mu_\pi(\mathscr{O}_\pi) = \dim V_\pi = d_\pi$) induced by the natural invariant symplectic form ω_π.

4. We define the coherent state $v_\theta := \pi(g_\theta)v_\pi \in V_\pi$ for $\theta \in \mathscr{O}_\pi$, which is uniquely determined by θ up to a phase, since $J_\pi(v_\theta) = \theta$.

5. For an operator $A \in \mathrm{End}(V_\pi)$, we have the *covariant* or *lower symbol*
$$L_A^\pi(\theta) := (v_\theta, Av_\theta)_{V_\pi}, \tag{3.131}$$
which uniquely determines A, since the coherent states $\{v_\theta\}_{\theta \in \mathscr{O}_\pi}$ are complete by means of the natural Kähler structure on $\mathscr{O}_\pi$ (cf.[8,53,55]). The lower symbol is covariant w.r.t. G:
$$L^\pi_{\pi(g)A\pi(g)^*}(\theta) = L_A^\pi(Ad_{g^{-1}}^*(\theta)),\ g \in G. \tag{3.132}$$

6. By duality and the Riesz-Fréchet theorem ($d_\pi < \infty$[56]), we obtain the *contravariant* or *upper symbol* $A \mapsto U_A^\pi$:
$$(A, B)_{HS} = \mathrm{tr}(A^*B) = \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ \overline{U_A^\pi(\theta)} L_A^\pi(\theta). \tag{3.133}$$

The normalisation of $d\mu_\lambda$ ensures that $U^\pi_{\mathbb{1}_{V_\pi}} = 1$. The upper symbol is covariant, as well:

$$U^\pi_{\pi(g)A\pi(g)^*}(\theta) = U^\pi_A(Ad^*_{g^{-1}}(\theta)),\ g \in G. \tag{3.134}$$

8. The map $U^\pi_A \mapsto L^\pi_A$ defines a positive G-invariant invertible operator K_π[57] on the finite dimensional space of functions $S_\pi := \mathrm{span}_\mathbb{C}\{U^\pi_A \mid A \in \mathrm{End}(V_\pi)\} \subset L^2(\mathscr{O}_\pi)$. G-invariance is to be understood w.r.t. the quasiregular representation $(\rho(g)f)(\theta) = f(Ad^*_{g^{-1}}(\theta)),\ f \in L^2(\mathscr{O}_\pi)$. Form the definition of lower and upper symbols, we infer that the kernel of K_π is determined by the overlap function of the coherent states:

$$L^\pi_A(\theta) = (v_\theta, Av_\theta)_{V_\pi} = \int_{\mathscr{O}_\pi} d\mu_\pi\ (\theta')|(v_\theta, v_{\theta'})_{V_\pi}|^2\ U^\pi_A(\theta'). \tag{3.135}$$

By means of K_π, the *Stratonovich-Weyl symbol* of A is defined to be:

$$W^\pi_A := K_\pi^{\frac{1}{2}} U^\pi_A = K_\pi^{-\frac{1}{2}} L^\pi_A, \tag{3.136}$$

where $K_\pi^{\frac{1}{2}}$ is the positive square root of K_π. It has the following properties:

(a) $\mathrm{End}(V_\pi) \ni A \mapsto W^\pi_A \in S_\pi$ is linear and bijective,

(b) $W^\pi_{A^*} = \overline{W^\pi_A}$,

(c) $W^\pi_{\mathbb{1}_{V_\pi}} = 1$,

(d) $W^\pi_{\pi(g)A\pi(g)^*} = \rho(g)(W^\pi_A)$,

(e)

$$\begin{aligned}(A, B)_{HS} &= \mathrm{tr}(A^* B) \\ &= \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ \overline{W^\pi_A(\theta)} W^\pi_B(\theta).\end{aligned} \tag{3.137}$$

9. Finally, the *Stratonovich-Weyl operator* $\Delta^\pi : \mathscr{O}_\pi \to \mathrm{End}(V_\pi)$ is constructed in spirit of the (Fourier-)Weyl elements, already familiar from the previous subsection (see (3.4)). Namely, we look for an operator valued function $\Delta^\pi = (\Delta^\pi)^*$ s.t.:

$$W^\pi_A(\theta) = \mathrm{tr}(\Delta^\pi(\theta)A) = (\Delta^\pi(\theta), A)_{HS}, \tag{3.138}$$

$$A = \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ W^\pi_A(\theta)\Delta^\pi(\theta) = (\Delta^\pi, W^\pi_A)_{L^2(\mathscr{O}_\pi)}.$$

To construct Δ^π, we decompose $S_\pi \cong \mathrm{End}(V_\pi) \cong V_\pi \otimes \overline{V_\pi} \cong \bigoplus_\eta V^{\oplus N^\eta_\pi}_{\eta \in \hat{G}} \cong \bigoplus_{\eta\in\hat{G}} \bigoplus_{s=1}^{N^\eta_\pi} V_\eta$[58], where $N^\eta_\pi \in \mathbb{N}_0$ is the multiplicity of η in $\pi \otimes \overline{\pi}$. Now, we can introduce the *generalised spin-weighted spherical harmonics* as an orthonormal basis of S_π:

$$\left(Y^\pi_{\eta s k}(g_\theta)_l\right)_{l=1,\dots,d_\eta} := (\eta_s(g_\theta)_{kl})_{l=1,\dots,d_\eta} \tag{3.139}$$

for $s = 1, \dots, N^\eta_\pi, k = 1, \dots, d_\eta, \theta \in \mathscr{O}_\pi$, where the matrix elements are computed w.r.t. to a basis, $\{v^{\eta,s}_i\}_{i=1}^{d_\eta}$, $v^{\eta,s}_{d_\eta} = v_{\eta,s}$, adapted to the weight decomposition $V_{\eta,s} = V_{\lambda_\eta} \oplus \bigoplus_\lambda V_\lambda$

of the s-th copy of V_η in $V_\pi \otimes \overline{V_\pi}$. We use the same notation for any adapted basis for a unitary irreducible representation of V of G. Here, λ_η denotes the highest weight of η, and v_η is a normalised weight vector of λ_η. The *generalised spherical harmonics* $Y^\pi_{(\eta,l_\eta)sk}(\theta) := Y^\pi_{(\eta,l_\eta)sk}(g_\theta)$ are obtained from the decomposition

$$\pi(g)_{id_\pi}\overline{\pi(g)}_{jd_\pi} = \sum_{\eta,l_\eta,s,k} C(\pi,i;\overline{\pi},j|\eta,k;s)\overline{C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta,0_{l_\eta};s)}Y^\pi_{(\eta,l_\eta)sk}(g), \tag{3.140}$$

since $v_\pi \otimes \overline{v}_\pi$ has (real) weight 0. The $C(\pi,m;\zeta,n|\eta,k;s)$'s denote the Clebsch-Gordan coefficients of the decomposition $\pi \otimes \zeta \cong \bigoplus_{\eta\in\hat{G}} \bigoplus_{s=1}^{N^\eta_{\pi,\zeta}} V_\eta$. These function on $\mathscr{O}_\pi$ (weight 0!) diagonalise the kernel K_π (cf.[8]),

$$\begin{aligned}
K_\pi(\theta,\theta') &= |(v_\theta,v_{\theta'})|^2 \\
&= |(\pi(g_\theta)v_\pi,\pi(g_{\theta'})v_\pi)|^2 \\
&= \Big|\sum_{i=1}^{d_\pi}\pi(g_{\theta'})_{id_\pi}\overline{\pi(g_\theta)}_{jd_\pi}\Big|^2 \\
&= \sum_{\substack{\eta,l_\eta,s,k\\ \eta',l'_{\eta'},s',k'}}\sum_{i,j=1}^{d_\pi} C(\pi,i;\overline{\pi},j|\eta,k;s)\overline{C(\pi,i;\overline{\pi},j|\eta',k';s')} \\
&\quad\times \overline{C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta,0_{l_\eta};s)}C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta',0_{l'_{\eta'}};s')Y^\pi_{(\eta,l_\eta)sk}(g_\theta)\overline{Y}^\pi_{(\eta',l'_{\eta'})s'k'}(g_{\theta'}) \\
&= \sum_{\eta,l_\eta,s,k} C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta,0_{l_\eta};s)^2 Y^\pi_{(\eta,l_\eta)sk}(\theta)\overline{Y}^\pi_{(\eta,l_\eta)sk}(\theta'),\ \theta,\theta'\in\mathscr{O}_\pi,
\end{aligned} \tag{3.141}$$

and thus determine the Stratonovich-Weyl operator Δ^π,

$$\begin{aligned}
\Delta^\pi(\theta) &= \sum_{\eta,l_\eta,s,k} C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta,0_{l_\eta};s)^{-1}Y^\pi_{(\eta,l_\eta)sk}(\theta)\int_{\mathscr{O}_\pi} d\mu_\pi(\theta')\overline{Y}^\pi_{(\eta,l_\eta)sk}(\theta')P_{\theta'} \\
&= K_\pi^{-\frac{1}{2}}P_\theta.
\end{aligned} \tag{3.142}$$

Here, $P_\theta = v_\theta \otimes v_\theta^*$ denotes the projection onto the coherent state v_θ. The phase convention for the Clebsch-Gordan coefficients is chosen s.t. $C(\pi,d_\pi;\overline{\pi},d_{\overline{\pi}}|\eta,0_{l_\eta};s) > 0$.

Remark III.31:

From (3.142), we see that the operator norm of $\Delta^\pi(\theta)$ is uniformly bounded in $\theta\in\mathscr{O}_\pi$. Therefore, the quantisation formula,

$$A_f := \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ f(\theta)\Delta^\pi(\theta), \tag{3.143}$$

defines an element of $\mathrm{End}(V_\pi)$ for any $f\in L^1(\mathscr{O}_\pi)$. Since the generalised spherical harmonics are smooth, (3.143) even makes sense for $f\in\mathscr{D}'(\mathscr{O}_\pi)$. By restricting to $f\in S_\pi\subset C^\infty(\mathscr{O}_\pi)$ the quantisation, $f\mapsto A_f$, becomes nondegenerate, but in contrast to $C^\infty(\mathscr{O}_\pi)$, which can be interpreted as the analog of the space $S^\infty_{\rho,\delta}$, S_π is not closed under multiplication.

Let us state the properties of the Stratonovich-Weyl quantisation (3.143) as a

Theorem III.32:

The Stratonovich-Weyl quantisation

$$Q_\varepsilon^{\mathrm{SW}}(f) := \int_{\mathscr{O}_\pi} d\mu_{\varepsilon^{-1}\pi}(\theta)\ f(\theta)\ \Delta^{\varepsilon^{-1}\pi}(\theta),\ Q_0^{\mathrm{SW}}(f) := f,\ \varepsilon^{-1} \in \mathbb{N}_0, \tag{3.144}$$

is a degenerate strict deformation[59] *quantisation of* $C^\infty(\mathscr{O}_\pi)$ *into* $\mathrm{End}(V_{\varepsilon^{-1}\pi})$ *in the sense of theorem III.14 (cf.*[7]*, Definition II.1.1.1.). Here,* $\varepsilon^{-1}\pi \in \hat{G}$ *is determined by the highest weight* $\varepsilon^{-1}\lambda_\pi \in \overline{C} \cap I_r$*, the intersection of the closed fundamental Weyl chamber* $\overline{C}$ *and the lattice of (real) integral weights* I_r^**. The Poisson structure on* $\mathscr{O}_\pi$ *is induced from the (minus) Lie-Poisson structure on* $\mathfrak{g}^*$*.*

Proof:

Degeneracy of the quantisation follows, because $\dim(\mathrm{End}(V_{\varepsilon^{-1}\pi})) < \infty$. For $f \in C^\infty(\mathscr{O}_\pi)$, we observe that the Stratonovich-Weyl quantisation $Q_\varepsilon^{\mathrm{SW}}(f)$ is related to Berezin quantisation $Q_\varepsilon^{\mathrm{B}}(f) = \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ f(\theta)\ P_\theta$ (cf.[7], Section III.1.11) by the operator $K_{\varepsilon^{-1}\pi}$:

$$Q_\varepsilon^{\mathrm{SW}}(f) = Q_\varepsilon^{\mathrm{B}}(K_{\varepsilon^{-1}\pi}^{\frac{1}{2}} f). \tag{3.145}$$

But, Landsman proves in[7], section II.1.11, that $Q_\varepsilon^{\mathrm{B}}$ is a strict quantisation of $C^\infty(\mathscr{O}_\pi)$. Although, we need to slightly correct the ε-expansion of $Q_\varepsilon^{\mathrm{B}}$, which is erroneous in[7]. Thus, if we controlled the ε-expansion of $K_{\varepsilon^{-1}\pi}^{\frac{1}{2}}$ to order ε, we would be able to decide the strictness of $Q_\varepsilon^{\mathrm{SW}}$ from the strictness of $Q_\varepsilon^{\mathrm{B}}$. To find the required ε-expansion of $K_{\varepsilon^{-1}\pi}^{\frac{1}{2}}$, we compute the ε-expansion of $K_{\varepsilon^{-1}\pi}$,

$$K_{\varepsilon^{-1}\pi} = K_\pi^{(0)} + \varepsilon K_\pi^{(1)} + O(\varepsilon^2), \tag{3.146}$$

and apply functional calculus, i.e.

$$\begin{aligned} K_{\varepsilon^{-1}\pi}^{\frac{1}{2}} &= \left(\mathrm{id}_{C(\mathscr{O}_\pi)} + \left(K_{\varepsilon^{-1}\pi} - \mathrm{id}_{C(\mathscr{O}_\pi)}\right)\right)^{\frac{1}{2}} \\ &= \mathrm{id}_{C(\mathscr{O}_\pi)} + \frac{1}{2}\left(K_{\varepsilon^{-1}\pi} - 1\right) + O(\varepsilon^2) = \mathrm{id}_{C(\mathscr{O}_\pi)} + \frac{1}{2}\varepsilon K_\pi^{(1)} + O(\varepsilon^2), \end{aligned} \tag{3.147}$$

since $\lim_{\varepsilon\to 0} K_{\varepsilon^{-1}\pi} = K_\pi^{(0)} = \mathrm{id}_{C(\mathscr{O}_\pi)}$ (cf.[7], Theorem III.1.11.1.), and $\forall f \in C(\mathscr{O}_\pi) : ||K_{\varepsilon^{-1}\pi} f||_\infty \leq ||f||_\infty$. Now, let us show that $K_{\varepsilon^{-1}\pi}$ actually has an expansion of the form (3.146). To this end, we analyse $K_{\varepsilon^{-1}\pi}$ in the form of (3.135):

$$\begin{aligned} \left(K_{\varepsilon^{-1}\pi} f\right)(\theta) &= \int_{\mathscr{O}_\pi} d\mu_{\varepsilon^{-1}\pi}(\theta')\ |(v_\theta, v_{\theta'})_{V_{\varepsilon^{-1}\pi}}|^2\ f(\theta') \\ &= \int_{\mathscr{O}_\pi} d\mu_{\varepsilon^{-1}\pi}(\theta')\ |(v_{\varepsilon^{-1}\pi}, (\varepsilon^{-1}\pi)(g_\theta^{-1} g_{\theta'}) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}}|^2\ f(\theta') \\ &= d_{\varepsilon^{-1}\pi} \int_{G_{\lambda_\pi}} dh \int_{G/G_{\lambda_\pi}} dg_{\theta'}\ |(v_{\varepsilon^{-1}\pi}, (\varepsilon^{-1}\pi)(g_\theta^{-1} g_{\theta'}) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}}|^2\ F(g_{\theta'} h) \\ &= d_{\varepsilon^{-1}\pi} \int_G dg\ |(v_{\varepsilon^{-1}\pi}, \pi(g_\theta^{-1} g) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}}|^2\ F(g) \end{aligned} \tag{3.148}$$

$$= d_{\varepsilon^{-1}\pi} \int_G dg\ |(v_{\varepsilon^{-1}\pi}, \pi(g) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}}|^2\ F(g_\theta g),$$

where we have used the (right) G_{λ_π}-invariance of $g \mapsto |(v_{\varepsilon^{-1}\pi}, \pi(g_\theta^{-1} g) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}}|^2$, as $v_{\varepsilon^{-1}\pi}$ is a highest weight vector. Here, $F = f \circ p \in C^\infty(G)$ is the (right) G_{λ_π}-invariant functions corresponding to $f \in C^\infty(\mathscr{O}_\pi)$ via $p : G \to G/G_{\lambda_\pi} \cong \mathscr{O}_\pi$. Again, exploiting the fact that $v_{\varepsilon^{-1}\pi}$ is a highest weight vector, we find:

$$\forall g \in G:\ (v_{\varepsilon^{-1}\pi}, \pi(g) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}} = (v_\pi, \pi(g) v_\pi)_{V_\pi}^{\varepsilon^{-1}}, \tag{3.149}$$

since the Cartan composite $V_{\lambda_1+\lambda_2}$ of two highest weights $\lambda_1, \lambda_2 \in \overline{C} \cap I_r^*$ has multiplicity 1 in $V_{\lambda_1} \otimes V_{\lambda_2}$ (cf.[19], VI.2.8). This allows us to write (3.148) in the form

$$\begin{aligned}
(K_{\varepsilon^{-1}\pi} f)(\theta) &= d_{\varepsilon^{-1}\pi} \int_G dg\ |(v_\pi, \pi(g) v_\pi)_{V_\pi}|^{2\varepsilon^{-1}}\ F(g_\theta g) \qquad (3.150)\\
&= d_{\varepsilon^{-1}\pi} \int_G dg\ e^{-2\varepsilon^{-1} S(g)}\ f(p(g_\theta g))\\
&= d_{\varepsilon^{-1}\pi} \int_{G/G_{\lambda_\pi}} dg_{\theta'}\ e^{-2\varepsilon^{-1} S(g_{\theta'})}\ f(p(g_\theta g_{\theta'}))\\
&= d_{\varepsilon^{-1}\pi} \int_{\mathscr{O}_\pi} d\mu_\pi(\theta')\ e^{-2\varepsilon^{-1} S_\pi(\theta')}\ f(g_\theta \cdot \theta')\\
&= d_{\varepsilon^{-1}\pi} \int_{U_{\lambda_\pi}} d\mu_\pi(\theta')\ e^{-2\varepsilon^{-1} S_\pi(\theta')}\ f(g_\theta \cdot \theta') + O(\varepsilon^\infty),
\end{aligned}$$

where $S(g) := -\log|(v_\pi, \pi(g) v_\pi)_{V_\pi}|$, which descends to S_π on $\mathscr{O}_\pi$ by (right) G_{λ_π}-invariance. The restriction to an arbitrary open neighbourhood U_{λ_π} of λ_π is justified by the fact that the positive function S_π assumes its sole absolute minimum at λ_π, $S_\pi(\lambda_\pi) = 0$, and the simple estimate:

$$\begin{aligned}
\left| d_{\varepsilon^{-1}\pi} \int_{\mathscr{O}_\pi \setminus U_{\lambda_\pi}} d\mu_\pi(\theta')\ e^{-2\varepsilon^{-1} S_\pi(\theta')}\ f(g_\theta \cdot \theta') \right| &\leq d_{\varepsilon^{-1}\pi} \int_{\mathscr{O}_\pi \setminus U_{\lambda_\pi}} d\mu_\pi(\theta')\ e^{-2\varepsilon^{-1} S_\pi(\theta')}\ |f(g_\theta \cdot \theta')| \qquad (3.151)\\
&\leq d_{\varepsilon^{-1}\pi} ||f(g_\theta \cdot (\,.\,))||_\infty e^{-2\varepsilon \inf_{\theta' \in \mathscr{O}_\pi \setminus U_{\lambda_\pi}} S_\pi(\theta')} \in O(\varepsilon^\infty),
\end{aligned}$$

as $d_{\varepsilon^{-1}\pi} \in O(\varepsilon^{-\frac{1}{2}\dim \mathscr{O}_\pi})$ by Weyl's formula (cf.[19], VI.1.7). Next, we choose U_{λ_π} small enough to identify it with a neighbourhood $W_{\lambda_\pi} \subset T_{\lambda_\pi}\mathscr{O}_\pi \cong \mathfrak{g} / \mathfrak{g}_{\lambda_\pi}$ of 0 via the exponential map. Due to the decomposition of $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ in to root spaces $\mathfrak{g}_{\pm\alpha}$, $\alpha \in R^+$, we have $\mathfrak{g}_{\lambda_\pi} = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in R^+ \\ \langle \lambda_\pi, \alpha \rangle_{\mathfrak{g}^*} = 0}} (\mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}))$, leading to complex coordinates $(z, \overline{z}) = \{(z_\alpha, \overline{z}_\alpha)\}_{\alpha \in R^+_{\lambda_\pi}}$ via

$$g_{\theta(z,\overline{z})} = \exp\left(\sum_{\alpha \in R^+_{\lambda_\pi}} (z_\alpha E_\alpha - \overline{z}_\alpha E_{-\alpha}) \right), \tag{3.152}$$

where we defined $R^+_{\lambda_\pi} := \{\alpha \in R^+ \mid \langle \lambda_\pi, \alpha \rangle_{\mathfrak{g}^*} \neq 0\}$, and introduced a Cartan-Weyl basis $\{H_i\}_{i=1}^r \subset \mathfrak{t}$, $\{E_\alpha, E_{-\alpha}\}_{\alpha \in R^+}$, $E_{\pm\alpha} \in \mathfrak{g}_\alpha$, for $\mathfrak{g}_\mathbb{C}$ (cf.[60]):

$$[H_i, H_j] = 0,\ i, j = 1, ..., r\ , \tag{3.153}$$

$$
\begin{aligned}
[H_i, E_{\pm\alpha}] &= \pm 2\pi i \alpha(H_i) E_{\pm\alpha},\ i = 1, ..., r\ ,\ \alpha \in R^+\ , \\
[E_\alpha, E_{-\alpha}] &= H_\alpha,\ \alpha \in R^+, \\
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta},\ \alpha, \beta \in R^+ \cup (-R^+),\ \alpha + \beta \neq 0,\ N_{\alpha,\beta} \neq 0 \text{ iff } \alpha + \beta \in R^+ \cup (-R^+).
\end{aligned}
$$

Here, $H_\alpha = \frac{2T_\alpha}{\langle T_\alpha, T_\alpha \rangle_{\mathfrak{g}}}$ is the co-root associated with $\alpha = \frac{1}{2\pi i} \langle T_\alpha,\ .\ \rangle_{\mathfrak{g}} \in R^+$[61].
Thus, we arrive at

$$
\begin{aligned}
(K_{\varepsilon^{-1}\pi} f)(\theta) = d_{\varepsilon^{-1}\pi} \int_{W_{\lambda_\pi} \subset \mathfrak{g}\,/\,\mathfrak{g}_\lambda} &\underbrace{\Big(\prod_{\alpha \in R^+_{\lambda_\pi}} \frac{dz_\alpha d\overline{z}_\alpha}{2\pi} \Big)}_{:= \sqrt{2\pi}^{-\dim(\mathscr{O}_\pi)} dz d\overline{z}} J(z, \overline{z}) e^{-2\varepsilon^{-1} S_\pi(z, \overline{z})} f(g_\theta \cdot (z, \overline{z})) \\
&+ O(\varepsilon^\infty),
\end{aligned} \tag{3.154}
$$

where J is the Jacobian associated with the exponential map. With (3.154) at hand, we are in a position to determine the ε-expansion of $K_{\varepsilon^{-1}\pi}$ to order ε by an appeal to Laplace's method (cf.[62]), i.e. we insert the Taylor expansion of $S_\pi(z, \overline{z}) = -\frac{1}{2} \log |(v_\pi, \pi(g_{\theta(z,\overline{z})}) v_\pi)_{V_\pi}|^2$ around $(z_{\lambda_\pi}, \overline{z}_{\lambda_\pi}) = 0$ up to fourth(!) order into (3.154), and invoke the unique extension of the unitary representation π to a holomorphic representation of $G_{\mathbb{C}}$:

$$
\begin{aligned}
&(v_\pi, \pi(g_{\theta(z,\overline{z})}) v_\pi)_{V_\pi} \\
&= \sum_{n=0}^\infty \frac{1}{n!} \Big(v_\pi, \Big(\sum_{\alpha \in R^+_{\lambda_\pi}} (z_\alpha d\pi(E_\alpha) - \overline{z}_\alpha d\pi(E_{-\alpha})) \Big)^n v_\pi \Big)_{V_\pi} \\
&= 1 + \sum_{\alpha \in R^+_{\lambda_\pi}} (z_\alpha \underbrace{(v_\pi, d\pi(E_\alpha) v_\pi)_{V_\pi}}_{=0} - \overline{z}_\alpha \underbrace{(v_\pi, d\pi(E_{-\alpha}) v_\pi)_{V_\pi}}_{=0}) \\
&\quad + \frac{1}{2} \sum_{\alpha, \beta \in R^+_{\lambda_\pi}} \Big(z_\alpha z_\beta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) v_\pi)_{V_\pi}}_{=0} + \overline{z}_\alpha \overline{z}_\beta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) v_\pi)_{V_\pi}}_{=0} \\
&\qquad - z_\alpha \overline{z}_\beta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) v_\pi)_{V_\pi}}_{= \delta_{\alpha,\beta} (v_\pi, d\pi(H_\alpha) v_\pi)_{V_\pi} = \delta_{\alpha,\beta} \langle \lambda_\pi, \alpha \rangle_{\mathfrak{g}^*} (\langle \alpha, \alpha \rangle_{\mathfrak{g}^*})^{-1}} - \overline{z}_\alpha z_\beta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) v_\pi)_{V_\pi}}_{=0} \Big) \\
&+ \frac{1}{6} \sum_{\alpha, \beta, \gamma \in R^+_{\lambda_\pi}} \Big(z_\alpha z_\beta z_\gamma \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_\gamma) v_\pi)_{V_\pi}}_{=0} - z_\alpha z_\beta \overline{z}_\gamma \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_{-\gamma}) v_\pi)_{V_\pi}}_{= \delta_{\alpha, \gamma-\beta} N_{\beta,-\gamma} \langle \lambda_\pi, \alpha \rangle_{\mathfrak{g}^*} (\langle \alpha, \alpha \rangle_{\mathfrak{g}^*})^{-1}} \\
&\quad - z_\alpha \overline{z}_\beta z_\gamma \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_\gamma) v_\pi)_{V_\pi}}_{=0} + z_\alpha \overline{z}_\beta \overline{z}_\gamma \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) v_\pi)_{V_\pi}}_{= \delta_{\alpha-\beta, \gamma} N_{\alpha,-\beta} \langle \lambda_\pi, \gamma \rangle_{\mathfrak{g}^*} (\langle \gamma, \gamma \rangle_{\mathfrak{g}^*})^{-1}} \\
&\quad - \overline{z}_\alpha z_\beta z_\gamma \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_\gamma) v_\pi)_{V_\pi}}_{=0} + \overline{z}_\alpha z_\beta \overline{z}_\gamma \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_{-\gamma}) v_\pi)_{V_\pi}}_{=0} \\
&\quad + \overline{z}_\alpha \overline{z}_\beta z_\gamma \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) d\pi(E_\gamma) v_\pi)_{V_\pi}}_{=0} - \overline{z}_\alpha \overline{z}_\beta \overline{z}_\gamma \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) v_\pi)_{V_\pi}}_{=0} \Big) \\
&+ \frac{1}{24} \sum_{\substack{\alpha, \beta, \\ \gamma, \zeta \in R^+_{\lambda_\pi}}} \Big(z_\alpha z_\beta z_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_\gamma) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&\qquad - z_\alpha \overline{z}_\beta z_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_\gamma) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0}
\end{aligned} \tag{3.155}
$$

$$
\begin{aligned}
&+ z_\alpha \overline{z}_\beta \overline{z}_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&- z_\alpha \overline{z}_\beta \overline{z}_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{\neq 0} \\
&+ z_\alpha \overline{z}_\beta z_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_\gamma) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{\neq 0} \\
&+ z_\alpha z_\beta \overline{z}_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_{-\gamma}) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{\neq 0} \\
&- z_\alpha z_\beta \overline{z}_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_{-\gamma}) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&- z_\alpha z_\beta z_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_\beta) d\pi(E_\gamma) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{\neq 0} \\
&+ \overline{z}_\alpha z_\beta z_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_\gamma) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&- \overline{z}_{-\alpha} \overline{z}_\beta z_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_\alpha) d\pi(E_{-\beta}) d\pi(E_\gamma) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&+ \overline{z}_\alpha \overline{z}_\beta \overline{z}_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&- \overline{z}_\alpha \overline{z}_\beta \overline{z}_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) d\pi(E_{-\gamma}) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{=0} \\
&+ \overline{z}_\alpha \overline{z}_\beta z_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_{-\beta}) d\pi(E_\gamma) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{=0} \\
&+ \overline{z}_\alpha z_\beta \overline{z}_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_{-\gamma}) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{=0} \\
&- \overline{z}_\alpha z_\beta \overline{z}_\gamma z_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_{-\gamma}) d\pi(E_\zeta) v_\pi)_{V_\pi}}_{=0} \\
&- \overline{z}_\alpha z_\beta z_\gamma \overline{z}_\zeta \underbrace{(v_\pi, d\pi(E_{-\alpha}) d\pi(E_\beta) d\pi(E_\gamma) d\pi(E_{-\zeta}) v_\pi)_{V_\pi}}_{=0} \Big) \\
&+ O((z, \overline{z})^5)
\end{aligned}
$$

where we repeatedly exploited the fact that v_π is a highest weight vector, the commutation relations (3.153) and the (unitary) representation π implements the adjointness relations $d\pi(E_\alpha)^* = d\pi(E_{-\alpha}),\ \alpha \in R^+$[63]. At this point it is important to note that non-zero contributions at order $(z, \overline{z})^3$ only arise for terms with $\alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma$ due to the geometry of root systems, i.e. the only multiples of a root α occurring in the decomposition of $\mathfrak{g}_\mathbb{C}$ are $\pm\alpha$. (3.155) and (15) imply:

$$
|(v_\pi, \pi(g_{\theta(z,\overline{z})}) v_\pi)_{V_\pi}|^2 = 1 - \sum_{\alpha \in R^+_{\lambda_\pi}} \frac{\langle \lambda_\pi, \alpha \rangle_{\mathfrak{g}^*}}{\langle \alpha, \alpha \rangle_{\mathfrak{g}^*}} z_\alpha \overline{z}_\alpha \tag{3.156}
$$

$$
\begin{aligned}
&+\frac{1}{6}\Bigg(\sum_{\substack{\alpha,\beta,\gamma\in R^+_{\lambda_\pi}\\ \alpha\neq\beta,\alpha\neq\gamma,\beta\neq\gamma}}\big(-z_\alpha z_\beta\overline{z}_\gamma(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_{-\gamma})v_\pi)_{V_\pi}\\
&\qquad\qquad+z_\alpha\overline{z}_\beta\overline{z}_\gamma(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_{-\gamma})v_\pi)_{V_\pi}\big)\\
&\underbrace{\qquad+\sum_{\substack{\alpha,\beta,\gamma\in R^+_{\lambda_\pi}\\ \alpha\neq\beta,\alpha\neq\gamma,\beta\neq\gamma}}\big(-\overline{z}_\alpha\overline{z}_\beta z_\gamma\overline{(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_{-\gamma})v_\pi)_{V_\pi}}+\overline{z}_\alpha z_\beta z_\gamma\overline{(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_{-\gamma})v_\pi)_{V_\pi}}\big)\Bigg)}_{=0,\text{ due to }d\pi(E_\alpha)^*=d\pi(E_{-\alpha}),\ \alpha\in R^+}\\
&+\frac{1}{12}\Bigg(3\sum_{\alpha,\beta\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}\frac{\langle\lambda_\pi,\beta\rangle_{\mathfrak{g}^*}}{\langle\beta,\beta\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha z_\beta\overline{z}_\beta\\
&+\sum_{\alpha,\beta,\gamma,\zeta\in R^+_{\lambda_\pi}}(-z_\alpha\overline{z}_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi})\\
&\qquad+z_\alpha\overline{z}_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\
&\qquad+z_\alpha z_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\
&\qquad-z_\alpha z_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\Bigg)\\
&+O((z,\overline{z})^5).
\end{aligned}\tag{3.157}
$$

Thus, we find by expanding the logarithm in S_π:

$$
\begin{aligned}
S_\pi(z,\overline{z})&=\frac{1}{2}\sum_{\alpha\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha\\
&\quad+\frac{1}{24}\Bigg(3\sum_{\alpha,\beta\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}\frac{\langle\lambda_\pi,\beta\rangle_{\mathfrak{g}^*}}{\langle\beta,\beta\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha z_\beta\overline{z}_\beta\\
&\qquad-\sum_{\alpha,\beta,\gamma,\zeta\in R^+_{\lambda_\pi}}\big(-z_\alpha\overline{z}_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi}\Bigg)\\
&\qquad\quad+z_\alpha\overline{z}_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\
&\qquad\quad+z_\alpha z_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\
&\qquad\quad-z_\alpha z_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\big)\\
&\quad+O((z,\overline{z})^5)\\
&=\frac{1}{2}\sum_{\alpha\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha+P_4(z,\overline{z})+O((z,\overline{z})^5).
\end{aligned}\tag{3.158}
$$

The integration in (3.154) can be extended from W_{λ_π} to $\mathfrak{g}\,/\,\mathfrak{g}_{\lambda_\pi}$ at the expense of a term of order ε^∞ by an estimate similar to (3.151), if the integrands are suitably continued beyond W_{λ_π}, which leads to ($\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}>0$ for $\alpha\in R^+_{\lambda_\pi}$):

$$
\begin{aligned}
&(K_{\varepsilon^{-1}\pi}f)\,(\theta)\\
&=\sqrt{2\pi}^{\,-\dim(\mathscr{O}_\pi)}d_{\varepsilon^{-1}\pi}\int_{\mathfrak{g}\,/\,\mathfrak{g}_\lambda}dz\;d\overline{z}\;J(z,\overline{z})e^{-2\varepsilon^{-1}S_\pi(z,\overline{z})}f(g_\theta\cdot(z,\overline{z}))+O(\varepsilon^\infty)
\end{aligned}\tag{3.159}
$$

$$
\begin{aligned}
&= \sqrt{2\pi}^{\,-\dim(\mathscr{O}_\pi)} d_{\varepsilon^{-1}\pi} \int_{\mathfrak{g}\,/\,\mathfrak{g}_\lambda} dz\, d\overline{z}\, J(z,\overline{z}) e^{-\varepsilon^{-1}\left(\sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}} z_\alpha \overline{z}_\alpha + 2P_4(z,\overline{z}) + O((z,\overline{z})^5)\right)} f(g_\theta\cdot(z,\overline{z})) \\
&\quad + O(\varepsilon^\infty) \\
&= \sqrt{\frac{\varepsilon}{2\pi}}^{\dim(\mathscr{O}_\pi)} d_{\varepsilon^{-1}\pi} \int_{\mathfrak{g}\,/\,\mathfrak{g}_\lambda} dz\, d\overline{z}\, J(\sqrt{\varepsilon}z,\sqrt{\varepsilon}\overline{z}) e^{-\left(\sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}} z_\alpha \overline{z}_\alpha + 2\varepsilon P_4(z,\overline{z})\right)} f(g_\theta\cdot(\sqrt{\varepsilon}z,\sqrt{\varepsilon}\overline{z})) \\
&\quad + O(\varepsilon^3) \\
&= \sqrt{\frac{\varepsilon}{2}}^{\dim(\mathscr{O}_\pi)} d_{\varepsilon^{-1}\pi} \left(\prod_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg(J(0,0) f(g_\theta\cdot(0,0)) \\
&\quad + \varepsilon\left(\sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}} \partial_{z_\alpha}\partial_{\overline{z}_\alpha}(Jf(g_\theta\cdot(\ .\ )))(0,0) + J(0,0)f(g_\theta\cdot(0,0))C_4(\lambda_\pi)\right)\Bigg) + O(\varepsilon^2) \\
&= \left(\prod_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{2\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg(J(0,0) f(g_\theta\cdot(0,0)) + \varepsilon\Bigg(\sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}} \partial_{z_\alpha}\partial_{\overline{z}_\alpha}(Jf(g_\theta\cdot(\ .\ )))(0,0) \\
&\qquad + J(0,0) f(g_\theta\cdot(0,0))\left(C_4(\lambda_\pi) + \sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg)\Bigg) + O(\varepsilon^2).
\end{aligned}
$$

Here, $C_4(\lambda_\pi)$ results from contribution of $e^{-2\varepsilon P_4(z,\overline{z})}$ to the order ε. The half-integer orders of ε coming from the Taylor expansions of J and $f(g_\theta\cdot(\ .\ ))$ vanish, because

$$
\int_{\mathfrak{g}\,/\,\mathfrak{g}_{\lambda_\pi}} dz\ d\overline{z}\ e^{-\sum_{\alpha\in R^+_{\lambda_\pi}} z_\alpha\overline{z}_\alpha} z^\beta\overline{z}^\gamma = \sqrt{\pi}^{\dim(\mathscr{O}_\pi)}\beta!\delta_{\beta,\gamma}, \quad \beta,\gamma\in\mathbb{N}_0^{\frac{1}{2}\dim(\mathscr{O}_\pi)}. \tag{3.160}
$$

Furthermore, we have

$$
J(0,0) = \left(\prod_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{2\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}\right)^{-1}, \tag{3.161}
$$

as we know that $\lim_{\varepsilon\to 0} K_{\varepsilon^{-1}\pi} = \mathrm{id}_{C(\mathscr{O}_\pi)}$. This allows us to (3.159) into a slightly simpler form:

$$
\begin{aligned}
(K_{\varepsilon^{-1}\pi}f)(\theta) &= f(\theta) + \varepsilon\Bigg(\left(\prod_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{2\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}\right)\sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}} \partial_{z_\alpha}\partial_{\overline{z}_\alpha}(Jf(g_\theta\cdot(\ .\ )))(0,0) \\
&\qquad + f(\theta)\left(C_4(\lambda_\pi) + \sum_{\alpha\in R^+_{\lambda_\pi}} \frac{\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg) + O(\varepsilon^2) \\
&= f(\theta) + \varepsilon\left(K^{(1)}_\pi f\right)(\theta) + O(\varepsilon^2).
\end{aligned} \tag{3.162}
$$

Now, following Landsman (cf.[7], Theorem III.1.11.4.), given any $\Phi_\varepsilon \in V_{\varepsilon^{-1}\pi}$, $||\Phi_\varepsilon|| = 1$, we have:

$$
(\Phi_\varepsilon, (Q^{\mathrm{SW}}_\varepsilon(f)Q^{\mathrm{SW}}_\varepsilon(f') - Q^{\mathrm{SW}}_\varepsilon(ff'))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} \tag{3.163}
$$

$$= (\Phi_\varepsilon, (Q^{\mathrm{B}}_\varepsilon(K^{\frac{1}{2}}_{\varepsilon^{-1}\pi} f) Q^{\mathrm{B}}_\varepsilon(K^{\frac{1}{2}}_{\varepsilon^{-1}\pi} f') - Q^{\mathrm{B}}_\varepsilon(K^{\frac{1}{2}}_{\varepsilon^{-1}\pi}(ff')))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}}$$
$$= (\Phi_\varepsilon, (Q^{\mathrm{B}}_\varepsilon(f) Q^{\mathrm{B}}_\varepsilon(f') - Q^{\mathrm{B}}_\varepsilon(ff'))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}}$$
$$+ \frac{\varepsilon}{2}(\Phi_\varepsilon, (Q^{\mathrm{B}}_\varepsilon(K^{(1)}_\pi f) Q^{\mathrm{B}}_\varepsilon(f') + Q^{\mathrm{B}}_\varepsilon(f) Q^{\mathrm{B}}_\varepsilon(K^{(1)}_\pi f') - Q^{\mathrm{B}}_\varepsilon(K^{(1)}_\pi(ff')))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} + O(\varepsilon^2)$$

for all $f, f' \in C^\infty(\mathscr{O}_\pi, \mathbb{R})$.
At this point, it is important to recall the equality $||A|| = \sup_{||\Phi_\varepsilon||=1} |(\Phi_\varepsilon, A\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}}|$ for $A \in \mathrm{End}(V_{\varepsilon^{-1}\pi}),\ A^* = A$. The terms in $O(\varepsilon)$ satisfy bounds of the form $C\varepsilon^k ||f||_{\infty,m} ||f'||_{\infty,n} ||\Phi_\varepsilon||^2$ for $k \in \mathbb{N},\ m, n \in \mathbb{N}_0,\ C > 0$, which will imply Dirac's condition (see theorem III.14) in the proof of strictness of $Q^{\mathrm{SW}}_\varepsilon$, if $Q^{\mathrm{B}}_\varepsilon$ is strict. Similarly, Rieffel's and von Neumann's conditions will follow from the strictness of $Q^{\mathrm{B}}_\varepsilon$ due to:

$$\begin{aligned}&(\Phi_\varepsilon, Q^{\mathrm{SW}}_\varepsilon(f)\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} \\ &= (\Phi_\varepsilon, Q^{\mathrm{B}}_\varepsilon(f)\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} + \varepsilon(\Phi_\varepsilon, Q^{\mathrm{B}}_\varepsilon(K^{(1)}_\pi f)\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} + O(\varepsilon^2), \\ &(\Phi_\varepsilon, (\tfrac{i}{\varepsilon}[Q^{\mathrm{SW}}_\varepsilon(f), Q^{\mathrm{SW}}_\varepsilon(f')] - Q^{\mathrm{SW}}_\varepsilon(\{f, f'\}_-))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} \\ &= (\Phi_\varepsilon, (\tfrac{i}{\varepsilon}[Q^{\mathrm{B}}_\varepsilon(f), Q^{\mathrm{B}}_\varepsilon(f')] - Q^{\mathrm{B}}_\varepsilon(\{f, f'\}_-))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} + O(\varepsilon).\end{aligned} \tag{3.164}$$

Finally, the strictness of $Q^{\mathrm{B}}_\varepsilon$ can be concluded from Landsman's argument subject to some minor modifications. More precisely, Landsman considers the first term in the last line of (3.163) in the form:

$$\begin{aligned}&(\Phi_\varepsilon, (Q^{\mathrm{B}}_\varepsilon(f) Q^{\mathrm{B}}_\varepsilon(f') - Q^{\mathrm{B}}_\varepsilon(ff'))\Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} \\ &= d_{\varepsilon^{-1}\pi} \int_G dg\ F(g)(\Phi_\varepsilon, (\varepsilon^{-1}\pi)(g) v_{\varepsilon^{-1}\pi})_{V_{\varepsilon^{-1}\pi}} I_\varepsilon(g),\end{aligned} \tag{3.165}$$

$$I_\varepsilon(g) := d_{\varepsilon^{-1}\pi} \int_G dh\ (v_\pi, \pi(h) v_\pi)^{\varepsilon^{-1}}_{V_\pi} F_{\varepsilon^{-1}\pi}(g, h),$$
$$F'_{\varepsilon^{-1}\pi}(g, h) := ((\varepsilon^{-1}\pi)(gh) v_{\varepsilon^{-1}\pi}, \Phi_\varepsilon)_{V_{\varepsilon^{-1}\pi}} (F'(gh) - F'(g)),$$

and then subjects $I_\varepsilon(g)$ to an asymptotic expansion by Laplace's method analogous to that of $(K_{\varepsilon^{-1}\pi} f)(\theta)$. In contrast to the previous calculation, neither I_ε nor $F'_{\varepsilon^{-1}\pi}$ are (right) G_{λ_π}-invariant, but are only (right) G_{λ_π}-equivariant:

$$\begin{aligned}F'_{\varepsilon^{-1}\pi}(g, hg_{\lambda_\pi}) &= e^{-\frac{i}{\varepsilon}\phi(g_{\lambda_\pi})} F'_{\varepsilon^{-1}\pi}(g, h), \quad F'_{\varepsilon^{-1}\pi}(gg_{\lambda_\pi}, h) = e^{-\frac{i}{\varepsilon}\phi(g_{\lambda_\pi})} F'_{\varepsilon^{-1}\pi}(g, g_{\lambda_\pi} h g^{-1}_{\lambda_\pi}), \\ I_\varepsilon(gg_{\lambda_\pi}) &= e^{-\frac{i}{\varepsilon}\phi(g_{\lambda_\pi})} I_\varepsilon(g), \qquad g_{\lambda_\pi} \in G_{\lambda_\pi},\ \phi : G_{\lambda_\pi} \to \mathbb{R}\end{aligned} \tag{3.166}$$

which ensures the invariance of (3.165). Nonetheless, the functions $h \mapsto (v_\pi, \pi(h) v_\pi)^{\varepsilon^{-1}}_{V_\pi} F_{\varepsilon^{-1}\pi}(g, h)$ are (right) G_{λ_π}-invariant for every $g \in G$. By the same arguments used in (3.150), (3.151) and (3.154), as $|F'_{\varepsilon^{-1}\pi}(g, h)| \leq 2||f'||_\infty$, we have

$$\begin{aligned}I_\varepsilon(g) = \sqrt{2\pi}^{\,-\dim(\mathscr{O}_\pi)} d_{\varepsilon^{-1}\pi} \int_{\mathfrak{g}\,/\,\mathfrak{g}_{\lambda_\pi}} dz\ d\overline{z}\ J(z, \overline{z})\ e^{-\varepsilon(-\log(v_\pi, \pi(g_{\theta(z,\overline{z})}) v_\pi)_{V_\pi})} F'_{\varepsilon^{-1}\pi}(g, g_{\theta(z,\overline{z})}) \\ + O(\varepsilon^\infty).\end{aligned} \tag{3.167}$$

Employing (3.155) and (15), we find:

$$
\begin{aligned}
&-\log(v_\pi,\pi(g_{\theta(z,\overline{z})})v_\pi)_{V_\pi} \qquad (3.168)\\
&= -\big((v_\pi,\pi(g_{\theta(z,\overline{z})})v_\pi)_{V_\pi}-1\big)-\frac{1}{2}\left((v_\pi,\pi(g_{\theta(z,\overline{z})})v_\pi)_{V_\pi}-1\right)^2+O((z,\overline{z})^5)\\
&= \frac{1}{2}\sum_{\alpha\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha-\frac{1}{6}\sum_{\substack{\alpha,\beta,\gamma\in R^+_{\lambda_\pi}\\ \alpha\neq\beta,\alpha\neq\gamma,\beta\neq\gamma}}\begin{aligned}\big(&-z_\alpha z_\beta\overline{z}_\gamma\delta_{\alpha,\gamma-\beta}N_{\beta,-\gamma}\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}(\langle\alpha,\alpha\rangle_{\mathfrak{g}^*})^{-1}\\ &+z_\alpha\overline{z}_\beta\overline{z}_\gamma\delta_{\alpha-\beta,\gamma}N_{\alpha,-\beta}\langle\lambda_\pi,\gamma\rangle_{\mathfrak{g}^*}(\langle\gamma,\gamma\rangle_{\mathfrak{g}^*})^{-1}\big)\end{aligned}\\
&\quad+\frac{1}{24}\Bigg(3\sum_{\alpha,\beta\in R^+_{\lambda_\pi}}\frac{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}\frac{\langle\lambda_\pi,\beta\rangle_{\mathfrak{g}^*}}{\langle\beta,\beta\rangle_{\mathfrak{g}^*}}z_\alpha\overline{z}_\alpha z_\beta\overline{z}_\beta\\
&\quad-\sum_{\alpha,\beta,\gamma,\zeta\in R^+_{\lambda_\pi}}\begin{aligned}\big(&-z_\alpha\overline{z}_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\ &+z_\alpha\overline{z}_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_{-\beta})d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\ &+z_\alpha z_\beta\overline{z}_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_{-\gamma})d\pi(E_{-\zeta})v_\pi)_{V_\pi}\\ &-z_\alpha z_\beta z_\gamma\overline{z}_\zeta(v_\pi,d\pi(E_\alpha)d\pi(E_\beta)d\pi(E_\gamma)d\pi(E_{-\zeta})v_\pi)_{V_\pi}\big)\end{aligned}\Bigg)\\
&\quad+O((z,\overline{z})^5),
\end{aligned}
$$

which yields the corrected expansion of $I_\varepsilon(g)$ to order ε:

$$
\begin{aligned}
I_\varepsilon(g) &= \left(\prod_{\alpha\in R^+_{\lambda_\pi}}\frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg(J(0,0)\underbrace{F'_{\varepsilon^{-1}\pi}(g,e)}_{=0} \qquad (3.169)\\
&\qquad+\varepsilon\Bigg(\sum_{\alpha\in R^+_{\lambda_\pi}}2\frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}(JF'_{\varepsilon^{-1}\pi}(g,g_{\theta(\,.\,,\,.\,)}))(0,0)\\
&\qquad+J(0,0)\underbrace{F'_{\varepsilon^{-1}\pi}(g,e)}_{=0}\left(C_3(\lambda_\pi)+2C_4(\lambda_\pi)+\sum_{\alpha\in R^+_{\lambda_\pi}}\frac{\langle\delta,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\right)\Bigg)\Bigg)+O(\varepsilon^2)\\
&= \varepsilon\sqrt{2}^{\,\dim(\mathscr{O}_\pi)}J(0,0)^{-1}\left(\sum_{\alpha\in R^+_{\lambda_\pi}}2\frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}(JF'_{\varepsilon^{-1}\pi}(g,g_{\theta(\,.\,,\,.\,)}))(0,0)\right)+O(\varepsilon^2)\\
&\underset{\substack{(\partial_{z_\alpha}J)(0,0)=0\\(\partial_{\overline{z}_\alpha}J)(0,0)=0}}{=}\varepsilon\sqrt{2}^{\,\dim(\mathscr{O}_\pi)}\left(\sum_{\alpha\in R^+_{\lambda_\pi}}2\frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}(F'_{\varepsilon^{-1}\pi}(g,g_{\theta(\,.\,,\,.\,)}))(0,0)\right)+O(\varepsilon^2)
\end{aligned}
$$

The terms of order $(z,\overline{z})^3$ do not yield contributions containing first derivatives of $F'_{\varepsilon^{-1}\pi}$, because of the constraints $\alpha\neq\beta,\ \alpha\neq\gamma,\ \beta\neq\gamma$ and (3.160). Clearly, the expansion is compatible with (right) G_{λ_π}-equivariance, as can be seen from (3.166) and the G_{λ_π}-invariance of the differential operator:

$$
\Delta_{\lambda_\pi} := \left(\sum_{\alpha\in R^+_{\lambda_\pi}}2\frac{\langle\alpha,\alpha\rangle_{\mathfrak{g}^*}}{\langle\lambda_\pi,\alpha\rangle_{\mathfrak{g}^*}}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}\right)_{|z=0,\overline{z}=0}. \qquad (3.170)
$$

Now, strictness of $Q^{\mathrm{B}}_{\varepsilon}$ follows from Landsman's argument. $\square$

Remark III.33:

The Stratonovich-Weyl quantisation on coadjoint orbits can be interpreted as the analog of Weyl quantisation on $\mathbb{R}^n$. In view of (3.137) it is distinguished from Berezin quantisation by the "tracial property" leading to the Stratonovich-Weyl-Fourier transform as pointed out by Figueroa, Gracia-Bondía and Várilly[8].

Now, we are in a position to define the Stratonovich-Weyl-Fourier transform:

Definition III.34 (cf.[8]):

The Stratonovich-Weyl-Fourier transform *is the composition of the Fourier transform $\mathscr{F} : L^2(G) \subset L^1(G) \to L^2(\hat{G})$ with the Stratonovich-Weyl symbol map $W : \hat{G} \to L^2(\bigcup_{\pi\in\hat{G}} \mathscr{O}_\pi) =: L^2(\mathscr{O}_G)$, i.e.:*

$$\begin{aligned}\mathscr{F}_{\mathrm{SW}}[\Psi](\pi,\theta) &= \hat{\Psi}_{\mathrm{SW}}(\pi,\theta) \\ &= W^\pi_{\mathscr{F}[\Psi]}(\theta) \\ &= \int_G dg\ \Psi(g)\,\mathrm{tr}(\Delta^\pi(\theta)\pi(g)) \\ &= \int_G dg\ \Psi(g)E(g;\pi,\theta),\ \ \Psi\in L^1(G),\ \ \pi\in\hat{G},\ \ \theta\in\mathscr{O}_\pi,\end{aligned} \tag{3.171}$$

where we introduced the integral kernel $E(g;\pi,\theta) = \mathrm{tr}(\Delta^\pi(\theta)\pi(g)) = W^\pi_{\pi(g)}(\theta)$. The space of integral coadjoint orbits*, $\mathscr{O}_G$, is endowed with the integral:*

$$\int_{\mathscr{O}_G} d\mu_{\mathscr{O}_G}(\pi,\theta)\Phi(\pi,\theta) := \sum_{\pi\in\hat{G}} d_\pi \int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ \Phi(\pi,\theta),\ \ \Phi\in L^1(\mathscr{O}_G). \tag{3.172}$$

It follows from the next theorem that the Stratonovich-Weyl-Fourier transform is well-defined and isometric from $L^2(G)$ to $L^2(\mathscr{O}_G)$ (the range is $\bigoplus_{\pi\in\hat{G}} S_\pi$). Moreover, it intertwines the convolution product and the *twisted product* coming from:

$$W^\pi_{AB}(\theta) = (W^\pi_A \star W^\pi_B)(\theta) = \int_{\mathscr{O}_\pi} d\mu_\pi(\theta')\int_{\mathscr{O}_\pi} d\mu_\pi(\theta'')\,\mathrm{tr}(\Delta^\pi(\theta)\Delta^\pi(\theta')\Delta^\pi(\theta''))W^\pi_A(\theta')W^\pi_B(\theta''). \tag{3.173}$$

Theorem III.35 (cf[8], Theorem 5):

The Stratonovich-Weyl-Fourier transform satisfies the inversion formula

$$\Psi(g) = \int_{\mathscr{O}_G} d\mu_{\mathscr{O}_G}(\pi,\theta)\overline{E}(g;\pi,\theta)\mathscr{F}_{\mathrm{SW}}[\Psi](\pi,\theta), \tag{3.174}$$

and the Parseval-Plancherel identity

$$\int_G dg\ |\Psi(g)|^2 = \int_{\mathscr{O}_G} d\mu_{\mathscr{O}_G}(\pi,\theta)\ |\mathscr{F}_{\mathrm{SW}}[\Psi](\pi,\theta)|^2. \tag{3.175}$$

Furthermore, the convolution product $*$ on $L^1(G)$ is intertwined with the twisted product $\star$:

$$\mathscr{F}_{\mathrm{SW}}[\Psi * \Phi] = \mathscr{F}_{\mathrm{SW}}[\Psi] \star \mathscr{F}_{\mathrm{SW}}[\Phi]. \tag{3.176}$$

The integral kernel $E : G \times \mathscr{O}_G \to \mathbb{C}$ has important properties that entail its independence of the representative in $\pi \in \hat{G}$ (property 2).

Proposition III.36 (cf.[8], Theorem 4):
The integral kernel E satisfies:

1. $\overline{E}(g;\pi,\theta) = E(g^{-1};\pi,\theta)$,
2. $E(\alpha_h(g);\pi,\theta) = E(g;\pi,Ad^*_{h^{-1}}(\theta))$,
3. $\int_{\mathscr{O}_\pi} d\mu_\pi(\theta)\ E(g;\pi,\theta) = \mathrm{tr}(\pi(g)) = \chi_\pi(g)$,
4. $\int_G dg\ E(g;\pi,\theta)\overline{E}(g;\pi,\theta') = d_\pi^{-1} I_\pi(\theta,\theta')$,
5. $(E(g;\ .\ ) \star \mathscr{F}_{\mathrm{SW}}[\Psi])(\pi,\theta) = \mathscr{F}_{\mathrm{SW}}[U_g\Psi](\pi,\theta),\ \Psi \in L^2(G)$,
6. $E(g;\ .\ ) \star E(h;\ .\ ) = E(gh;\ .\ )$.

Remark III.37:
Property 5 & 6 of proposition III.36 tell us that the action of the exponentials $U_g, g \in G$, of the "momenta" $P_X, X \in \mathfrak{g}$, in (3.1) is turned in to (non-commutative) multiplication with the functions $E(g;\ .\ ),\ g \in G$ by the Stratonovich-Weyl-Fourier transform. Thus, the range of the latter qualifies as (non-commutative) flux representation for loop quantum gravity in the terminology of[64].

As mentioned in the beginning of this subsection, the Stratonovich-Weyl-Fourier transform allows us to study the effect of scaling in the global Fourier correspondence for compact Lie groups. To this end, we distinguish to types of ε-scaled Stratonovich-Weyl-Fourier transforms ($\varepsilon = \frac{1}{k},\ k \in \mathbb{N}$):

1. ("position" scaling): $\mathscr{F}^{\varepsilon}_{\mathrm{SW}}[\Phi](\pi,\theta) := \int_G dg\ E(g^{\frac{1}{\varepsilon}};\pi,\theta)\Psi(g)$,
2. ("momentum" scaling): $\mathscr{F}_{\mathrm{SW},\varepsilon}[\Phi](\pi,\theta) := \int_G dg\ E(g;\varepsilon^{-1}\pi,\theta)\Psi(g)$,
   where $\varepsilon^{-1}\pi$ is the unitary irreducible representation of G with highest weight $\varepsilon^{-1}\lambda_\pi$.

As above, the restriction to discrete scalings, $\varepsilon = \frac{1}{k},\ k \in \mathbb{N}$, is necessary as otherwise the powers $g^{\frac{1}{\varepsilon}}$ would not be uniquely defined, and $\varepsilon^{-1}\lambda_\pi$ would not belong to $\overline{C} \cap I^*_r$. Clearly, these transformations coincide for commutative Lie groups, where the Stratonovich-Weyl-Fourier transform coincides with usual Fourier transform. Structurally, the "momentum" scaled transform seems to be favoured, as it can be expressed in terms of the unscaled transform:

$$\mathscr{F}_{\mathrm{SW},\varepsilon}[\Phi](\pi,\theta) = \int_G dg\ E(g;\varepsilon^{-1}\pi,\theta)\Psi(g) = \mathscr{F}_{\mathrm{SW}}[\Psi](\varepsilon^{-1}\pi,\theta). \tag{3.177}$$

In contrast, the "position" scaled transform is not related to unscaled transform, because the ε-th power is not a diffeomorphism of G (unless $\varepsilon = 1$). Therefore, we stick to the "momentum" scaled transform in the following. Unfortunately, we immediately recognize, that the scaled transform no longer defines an invertible map, as it restricts the Stratonovich-Weyl-Fourier transform to those integral coadjoint orbits associated with the sub-lattice $\varepsilon^{-1}I^*_r \subset I^*_r$. Again, this is a feature forced

upon us by the rigidity of compact Lie groups. In subsection III D, we will discuss a possible way to remove this restriction for $G = U(1)$.

Nonetheless, the Stratonovich-Weyl-Fourier transform provides us with a ε-scaled transform for systems modelled on integral coadjoint orbits $\mathscr{O}_\pi$, which was already exploited in[1] for $G = SU(2)$. To see how this works in the general case, we note that λ_π and $\varepsilon^{-1}\lambda_\pi$ have the same, possibly degenerate, orbit type $\mathscr{O}_\pi$, since $G_{\lambda_\pi} = G_{\varepsilon^{-1}\lambda_\pi}$. Thus, we may study the "semiclassical" limit, $\varepsilon \to 0$, of sequences of operators $\{A_\varepsilon\}_{\varepsilon^{-1}\in\mathbb{N}}$, $A_\varepsilon \in \mathrm{End}(V_{\varepsilon^{-1}\pi})$, in terms of their Stratonovich-Weyl symbols $W^{\varepsilon^{-1}\pi}_{A_\varepsilon} \in S_{\varepsilon^{-1}\pi} \subset C^\infty(\mathscr{O}_\pi) \subset L^2(\mathscr{O}_\pi)$ and the twisted products $\star_\varepsilon$, e.g.

$$A_\varepsilon = i\varepsilon \sum_{i=1}^n B_i d(\varepsilon^{-1}\pi)(\tau_i) \;\Leftrightarrow\; W^{\varepsilon^{-1}\pi}_{A_\varepsilon}(\theta) = i\varepsilon \sum_{i=1}^n B_i \frac{d}{dt}_{|t=0} E(e^{t\tau_i};\varepsilon^{-1}\pi,\theta), \tag{3.178}$$

which is a generalisation of the magnetic part of the Pauli Hamiltonian to G. Here, $\{\tau_i\}_{i=1}^n$ is a basis of $\mathfrak{g}$ and $B = (B_i)_{i=1,..,n} \in \mathbb{R}^n$. Another physical application, apart from spin-orbit coupling discussed in[1,65], where these structures feature prominently, is the description of classical and quantum particles with internal symmetry in external gauge fields, which are governed by the classical respectively quantum *Wong equations* (cf.[7,28]), and its relation to *quantum chaos* (cf.[66–69]).

If we wanted to avoid working with degenerate coadjoint orbits, i.e. those corresponding to singular integral weights λ, $\langle\alpha,\lambda\rangle_{\mathfrak{g}^*} = 0$ for some $\alpha \in R^+$, we could incorporate the usual shift by half the sum of the positive roots, $\delta = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, in to the correspondence between dominant integral weights and integral coadjoint orbits,

$$\lambda_\pi \to \mathscr{O}^\delta_\pi := \{\theta \in \mathfrak{g}^* \mid \exists g \in G : \theta = Ad^*_g(\lambda_\pi + \delta)\}, \tag{3.179}$$

which allows us to work solely with coadjoint orbits of strongly dominant integral weights, since

$$\lambda \in \overline{C} \cap I^*_r \Leftrightarrow \lambda + \delta \in C \cap I^*_r \tag{3.180}$$

(cf.[19,60]). The coadjoint orbits of strongly dominant integral weights are isomorphic to the simply connected generalised flag variety $G/T \cong G_{\mathbb{C}}/B^+$, where B^+ is the standard Borel subgroup associated with the positive roots R^+ (cf.[19,52]).

Another advantage of the Stratonovich-Weyl-Fourier transform is that it enables us to pass from matrix valued symbols on $G\times\hat{G}$ in the definition of the global pseudo-differential calculus of subsection (III A 1) to genuine functions on $G \times \mathscr{O}_G$. Namely, the Stratonovich-Weyl symbol of a symbol $\sigma \in \hat{\mathscr{D}}'(\hat{G}\times G)$ is just the Stratonovich-Weyl-Fourier transform of its left convolution kernel F_σ:

$$\begin{aligned}\sigma_{\mathrm{SW}}(\pi,\theta;g) &= W^\pi_{\sigma(\pi,g)}(\theta) = \mathrm{tr}(\Delta^\pi(\theta)\sigma(\pi,g)) \\ &= \int_G dh\; E(h;\pi,\theta)F_\sigma(h,g) = \hat{F}_{\mathrm{SW}}(\pi,\theta;g).\end{aligned} \tag{3.181}$$

In this way, the action of the symbol σ on $C^\infty(G)$ is related to the twisted product $\star$:

$$\left(\rho_{\mathrm{L}}(F_\sigma)\Psi\right)(g) = \int_{\mathscr{O}_G} d\mu_{\mathscr{O}_G}\;\left(\overline{E}(g;\;.\;)\star\sigma_{\mathrm{SW}}(\;.\;;g)\right)(\pi,\theta)\hat{\Psi}_{\mathrm{SW}}(\pi,\theta). \tag{3.182}$$

Also, the composition of two symbols σ, τ becomes expressible in terms of the twisted product (cp. proposition III.2, 6.):

$$\mathscr{F}_{\mathrm{SW}}[F_\sigma *_{\mathrm{L}} F_\tau](\pi,\theta;g) = \int_G dh\ F_\sigma(h,g)\left(E(h;\ .\ ) \star \tau_{\mathrm{SW}}(\ .\ ; h^{-1}g)\right)(\pi,\theta). \tag{3.183}$$

C. Coherent states & quantisation

In the previous subsection, we have encountered the concept of Berezin quantisation w.r.t. a (complete) system of coherent states (cf.[70,71]), and its close relation to Stratonovich-Weyl quantisation on coadjoint orbits. As explained in section II B, we have a (complete) system of coherent states for any compact Lie group G at our disposal, which naturally leads to the question how Berezin quantisation on T^*G in terms of these coherent states relates to the Kohn-Nirenberg and Weyl quantisation discussed in subsection III A. Moreover, it is important to understand, whether this Berezin quantisation of T^*G gives a suitable framework for the Born-Oppenheimer approximation and adiabatic perturbation theory. As we will argue, the equivalent roles played by Berezin and (Stratonovich-)Weyl quantisation on coadjoint orbits with regard to such a framework is quite special to the case of compact phase spaces, and therefore finite dimensional Hilbert spaces, while the existence of an analog of the (smoothing) operator $K_\pi^{\frac{1}{2}}$ (3.136) will be the reason, why Weyl quantisation is favoured, if the construction of a twisted product for phase space functions is intended.

Let us start by defining Berezin quantisation on T^*G in terms of the coherent states from section II B. To this end, we note that the coherent states $\Psi^t_z \in L^2(G),\ t \in \mathbb{R},\ z \in G_{\mathbb{C}}$, (2.14) provides us with a *coherent pure state quantisation* of T^*G (in the sense of[7]). Therefore, the following definition makes sense for small enough $t > 0$ (see (2.37), cf. also[53]):

Definition III.38:

Given $f \in L^p(T^*G, dg\ dX)$, $1 \le p \le \infty$, its Berezin quantisation is

$$Q^{\mathrm{B}}_t(f) := \int_{G_{\mathbb{C}}} dz\ \nu_t(z)\ f(\Phi^{-1}(z))\ |\Psi^t_z\rangle\langle\Psi^t_z|, \tag{3.184}$$

which defines an operator in $\mathscr{S}_p(L^2(G))$, the pth Schatten class in $\mathscr{B}(L^2(G))$, since $\mathrm{tr}\left(|Q^{\mathrm{B}}_t|^p\right) \le$ $||f||^p_p)$ *for $1 \le p < \infty$ and $||Q^{\mathrm{B}}_t(f)|| \le ||f||_\infty$. $f \circ \Phi^{-1}$ is called the* upper *or* contravariant symbol *of the operator $Q^{\mathrm{B}}_t(f)$. The* lower *or* covariant symbol *of an operator $A \in \mathscr{B}(L^2(G))$ is*

$$L^t_A(z,\overline{z}) := \frac{\langle\Psi^t_z|A|\Psi^t_z\rangle}{\langle\Psi^t_z|\Psi^t_z\rangle}, \tag{3.185}$$

which satisfies $||L^t_A||_\infty \le ||A||$, and $||L^t_A|^p||^p_p \le \mathrm{tr}\left(|A|^p\right)$ for $A \in \mathscr{S}_p(L^2(G))$.

Remark III.39:

Berezin quantisation would take a more natural form if the conjecture II.9 turned out to be true, as in that case the normalised projection $P_t(z,\overline{z})$ onto the coherent state vectors Ψ^t_z would provide

a resolution of unity w.r.t. to the Liouville measure on T^*G, and (3.184) would take the form:

$$Q_t^{\mathrm{B}}(f) = C_t \int_G \int_{\mathfrak{g}} dg\ dX\ f(g,X)\ P_t(ge^{iX}, ge^{-iX}). \tag{3.186}$$

Corollary III.40 (cf.[53], Appendix 1):
Berezin quantisation is real, $Q_t^{\mathrm{B}}(f)^* = Q_t^{\mathrm{B}}(\overline{f})$, and positive, $Q_t^{\mathrm{B}}(f) \geq 0$ if $f \geq 0$ a.e.. The upper and lower symbols associated with Berezin quantisation are dual to one another[72], i.e.

$$\mathrm{tr}(AQ_t^{\mathrm{B}}(f)) = \int_{G_{\mathbb{C}}} \frac{dz\ \nu_t(z)}{\langle \Psi_z^t | \Psi_z^t \rangle} L_A^t(z,\overline{z}) f(\Phi^{-1}(z)), \quad A \in \mathscr{S}_q(L^2(G)),\ f \in L^p(T^*G, dg\ dX), \tag{3.187}$$

for $\frac{1}{q} + \frac{1}{p} = 1$. Moreover, the normalised coherent state projections $P_t(z,\overline{z})$ are complete, i.e. $\mathrm{img}(Q_t^{\mathrm{B}})$ is sequentially strongly dense in $\mathscr{B}(L^2(G))$, due to the analyticity of the coherent states Ψ_z^t, as the latter implies $\forall A \in \mathscr{S}_1(L^2(G))$: $L_A^t = 0$ if and only if $A = 0$.

While the corollary tells us, that the upper symbol exists for a sequentially strongly dense subspace of operators in $\mathscr{B}(L^2(G))$, it does not ensure that all operators in $\mathscr{B}(L^2(G))$ can be obtained from Berezin quantisation, as $L^2(G)$ is infinite dimensional in contrast to the representation spaces V_π considered in the previous subsection. Therefore, the existence of an upper symbol for a product of Berezin quantisations $Q_t^{\mathrm{B}}(f)Q_t^{\mathrm{B}}(f')$ cannot be concluded. Nevertheless, we might wonder whether there exists a set of functions S_{B}^t on T^*G, such that on the one hand $Q_t^{\mathrm{B}} : S_{\mathrm{B}}^t \to \mathscr{B}(L^2(G))$ is nondegenerate, and $\forall f, f' \in S_{\mathrm{B}}^t : \exists f'' \in S_{\mathrm{B}}^t : Q_t^{\mathrm{B}}(f)Q_t^{\mathrm{B}}(f') = Q_t^{\mathrm{B}}(f'') =: Q_t^{\mathrm{B}}(f \star_t f')$, while on the other hand Berezin quantisation of S_{B}^t encompasses sufficiently many "interesting" operators, and $\star_t$ can be (asymptotically) expanded w.r.t. the Poisson bracket on T^*G. To analyse this question in some detail, we first consider the relation between upper and lower symbol, if both exist, of a given Berezin quantisation. Similar to (3.135), the lower symbol of $Q_t^{\mathrm{B}}(f)$ can be expressed in terms of the upper symbol and the overlap function of the coherent states:

$$L^t_{Q_t^{\mathrm{B}}(f)}(z,\overline{z}) = (\langle \Psi_z^t | \Psi_z^t \rangle)^{-1} \int_{G_{\mathbb{C}}} dz'\ \nu_t(z')\ |\langle \Psi_z^t | \Psi_{z'}^t \rangle|^2\ f(\Phi^{-1}(z')). \tag{3.188}$$

This expression is also valid in the case of standard coherent states for $\mathbb{R}^n$, where it takes the well-known explicit form (see the discussion at the beginning of subsection II B 1, cp.[7,11]):

$$L^t_{Q_t^{\mathrm{B}}(f)}(z,\overline{z}) = \int_{\mathbb{C}^n} \frac{d\Re(z)\ d\Im(z)}{(2\pi t)^n}\ e^{-\frac{1}{2t}(z-z')(\overline{z}-\overline{z}')}\ f(\Phi^{-1}(z)) = \left(e^{2t\partial_z \partial_{\overline{z}}} f \circ \Phi^{-1}\right)(z), \tag{3.189}$$

from which it can be inferred that $L^t_{Q_t^{\mathrm{B}}}$ is the restriction of an entire function on $\mathbb{C}^{2n}$ even if $f \in \mathscr{S}'(\mathbb{R}^{2n})$ due to the smoothing nature of $e^{2t\partial_z\partial_{\overline{z}}} = e^{\frac{t}{2}\Delta_{(q,p)}}$, $z = q + ip = \Phi(q,p)$. Moreover, Weyl quantisation (see (3.4)), $Q_t^{\mathrm{B}}(f) = A_\sigma$, fits into the picture in the same way as for the coadjoint orbits (3.136) (cp.[11]):

$$L^t_{Q_t^{\mathrm{B}}(f)}(q,p) = \left(e^{\frac{t}{4}\Delta_{(q,p)}}\sigma\right)(q,p), \qquad \sigma(q,p) = \left(e^{\frac{t}{4}\Delta_{(q,p)}} f\right)(q,p), \tag{3.190}$$

which tells us that Weyl symbols are typically better behaved than upper symbols. Especially, since we would expect the (asymptotic) expansion of a twisted product $\star_t$ to be determined by local products of derivatives of the factors, which would be problematic in case of distributional

symbols.

A similar situation is to be expected for a compact Lie group G, and the Berezin quantisation (3.184) of T^*G, as the latter is non-compact. We will show this explicitly for $G = U(1)$[73], or more precisely for $U(1)$-equivariant Berezin quantisation. The coherent states from section II B take the following explicit form for $G = U(1)$ (cp.[25]):

Considering the Hilbert spaces

$$\mathfrak{H}_{j_0} := \{\Psi \in L^2_{\text{loc}}(\mathbb{R}) \mid \forall j \in \mathbb{Z} : \Psi(\varphi + 2\pi j) = e^{2\pi i j_0 j}\Psi(\varphi)\},\ j_0 \in [0,1), \tag{3.191}$$

equipped with the scalar product

$$(\Psi_1, \Psi_2)_{j_0} := \int_{[0,2\pi)} \frac{d\varphi}{2\pi}\, \overline{\Psi_1(\varphi)}\Psi_2(\varphi), \tag{3.192}$$

we have the following orthonormal bases of eigenfunction of $J := -i\partial_\varphi,\ D(J) := \mathfrak{H}_{j_0} \cap H^1_{\text{loc}}(\mathbb{R})$ (subject to the boundary conditions associated with (3.191)):

$$\Psi^{j_0}_j \in \mathfrak{H}_{j_0} : \Psi^{j_0}_j(\varphi) := e^{i(j+j_0)\varphi} = \langle\varphi|j + j_0\rangle,\ j \in \mathbb{Z}\,. \tag{3.193}$$

Moreover, the Hilbert spaces $\mathfrak{H}_{j_0}$ are representation spaces of the $U(1)$-Weyl algebra (cp. (3.8))

$$\begin{aligned}
&(V_t(\beta)\Psi)(\varphi) := \Psi(\varphi + t\beta), (U_t(m)\Psi)(\varphi) := e^{im\varphi}\Psi(\varphi),\ \Psi \in \mathfrak{H}_{j_0}, \\
&V_t(\beta)U_t(m) = e^{itm\beta}U_t(m)V_t(\beta), \\
&U_t(m)U_t(n) = U_t(m+n),\ V_t(\beta)V_t(\gamma) = V_t(\beta+\gamma), \\
&U_t(0) = \mathbb{1} = V_t(0),\ U_t(m)^* = U_t(-m),\ V_t(\beta)^* = V_t(-\beta),\ m,n \in \mathbb{Z},\ \beta,\gamma \in \mathbb{R}\,.
\end{aligned} \tag{3.194}$$

In terms of the bases the (equivariant) coherent states are:

$$\begin{aligned}
\Psi^{t,j_0}_\xi \in \mathfrak{H}_{j_0} : \Psi^{t,j_0}_\xi(\varphi) &= \sum_{j\in\mathbb{Z}} (\xi e^{tj_0})^{-j} e^{-\frac{t}{2}j^2}\Psi^{j_0}_j(\varphi) \\
&= \langle\varphi|\xi, j_0\rangle_t
\end{aligned} \tag{3.195}$$

for $\xi \in \mathbb{C}^* = U(1)_{\mathbb{C}} \cong T^*U(1)$. These vectors are eigenfunctions of the *annihilation operators* $X_t := e^{-\frac{t}{2}}U_t(1)e^{-tJ}$,

$$X_t|\xi, j_0\rangle_t = \xi|\xi, j_0\rangle_t, \tag{3.196}$$

where the former can be obtained from the complexifier method, $X_t := e^{-\frac{t}{2}J^2}Ue^{\frac{t}{2}J^2}$ (cf.[21,74–76]). Now, given a Berezin quantisation $Q^{\text{B}}_t(f)$ of $f \in C^\infty_b(T^*U(1))$, the formula connecting the upper symbol $f \circ \Phi^{-1}$, $\Phi(e^{i\varphi}, l) = e^{i\varphi}e^{-l} = e^{-l+i\varphi}$, and the lower symbol $L^t_{Q^{\text{B}}_t(f)}$ is:

$$\begin{aligned}
L^t_{Q^{\text{B}}_t(f)}(\xi,\bar{\xi}) &= \int_{\mathbb{C}^*} \frac{d\xi' \wedge d\bar{\xi}'}{4\pi i\sqrt{\pi t}} f(\Phi^{-1}(\xi'))\ |\xi'|^{-2}\ e^{-\frac{(\log|\xi'|-j_0 t)^2}{t}} \frac{|{}_t\langle\xi, j_0|\xi', j_0\rangle_t|^2}{{}_t\langle\xi, j_0|\xi, j_0\rangle_t} \\
&= \int_{[0,2\pi)}\int_{\mathbb{R}} \frac{d\varphi'\ dl'}{2\pi\sqrt{\pi t}} f(\varphi', l')\ e^{-\frac{(l'-j_0 t)^2}{t}} \frac{|{}_t\langle\xi, j_0|\Phi(e^{i\varphi'}, l'), j_0\rangle_t|^2}{{}_t\langle\xi, j_0|\xi, j_0\rangle_t},
\end{aligned} \tag{3.197}$$

where we chose to base the Berezin quantisation on the resolution of unity (2.30). Putting $\xi = e^{-l+i\varphi}$ and using

$$
\begin{aligned}
{}_t\langle \Phi(e^{i\varphi}, l), j_0 | \Phi(e^{i\varphi'}, l'), j_0\rangle_t &= \sum_{j\in\mathbb{Z}} e^{-tj^2} e^{j((l+l')+i(\varphi-\varphi')-2j_0 t)} \\
&= \vartheta_3(\tfrac{i}{2\pi}(-(l+l')-i(\varphi-\varphi')+2j_0 t)|\tfrac{it}{\pi}) \\
&= \sqrt{\frac{\pi}{t}} e^{\frac{1}{4t}(-(l+l')-i(\varphi-\varphi')+2j_0 t)^2} \vartheta_3(\tfrac{1}{2t}(-(l+l')-i(\varphi-\varphi'))+j_0|\tfrac{i\pi}{t}) \\
&= \sqrt{\frac{\pi}{t}} \sum_{j\in\mathbb{Z}} e^{\frac{1}{4t}(((l-j_0 t)+(l'-j_0 t))+i(\varphi-\varphi')-2\pi i j)^2},
\end{aligned}
\tag{3.198}
$$

we have:

$$
\begin{aligned}
L^t_{Q^{\mathrm{B}}_t(f)}(\Phi(e^{i\varphi}, l)) &= \frac{e^{-\frac{(l-j_0 t)^2}{t}}}{\vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})} \int_{[0,2\pi)} \int_{\mathbb{R}} \frac{d\varphi'\ dl'}{2\pi t} f(\varphi', l')\ e^{-\frac{(l'-j_0 t)^2}{t}} \\
&\quad \times \sum_{j,k\in\mathbb{Z}} e^{\frac{1}{4t}(((l-j_0t)+(l'-j_0t))+i(\varphi-\varphi')-2\pi i j)^2} e^{\frac{1}{4t}(((l-j_0t)+(l'-j_0t))-i(\varphi-\varphi')+2\pi i k)^2} \\
&= \int_{[0,2\pi)} \int_{\mathbb{R}} \frac{d\varphi'\ dl'}{2\pi t\ \vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})}\ f(\varphi', l')\ e^{-\frac{1}{2t}((l-j_0t)-(l'-j_0t))^2} \\
&\quad \times \sum_{j,k\in\mathbb{Z}} e^{-\frac{1}{2t}(\varphi-\varphi'-2\pi j)^2} e^{-\frac{1}{4t}(2\pi(j-k))} e^{-\frac{1}{2t}(2\pi(j-k))(\varphi-\varphi'-2\pi j+i((l-j_0t)+(l'-j_0t)))} \\
&\underset{\substack{n=j-k \\ (SL_2(\mathbb{Z})\ \text{transform})}}{=} \int_{[0,2\pi)} \int_{\mathbb{R}} \frac{d\varphi'\ dl'}{2\pi t\ \vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})}\ f(\varphi', l')\ e^{-\frac{1}{2t}((l-j_0t)-(l'-j_0t))^2} \\
&\quad \times \sum_{j,n\in\mathbb{Z}} e^{-\frac{1}{2t}(\varphi-\varphi'-2\pi j)^2} e^{-\frac{\pi^2}{t}n^2} e^{-\frac{\pi}{t}n(\varphi-\varphi'-2\pi j+i((l-j_0t)+(l'-j_0t)))} \\
&= \sum_{n\in\mathbb{Z}} e^{-\frac{\pi^2}{t}n^2} e^{-2\pi i n(\frac{l}{t}-j_0)} \int_{\mathbb{R}} \frac{dl'}{2\pi t\, \vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})} \int_{[0,2\pi)} d\varphi'\, f(\varphi', l') \\
&\quad \times \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2t}((l-l'-i\pi n)^2+(\varphi-\varphi'+\pi n-2\pi j)^2)} \\
&= \underbrace{\sum_{n\in\mathbb{Z}} e^{-\frac{\pi^2}{t}n^2} e^{-2\pi i n(\frac{l}{t}-j_0)}}_{=\vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})} \int_{\mathbb{R}} \frac{dl'}{2\pi t\ \vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})} \int_{[0,2\pi)} d\varphi'\, f(\varphi', l') \\
&\quad \times \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2t}((l-l')^2+(\varphi-\varphi'+2\pi j)^2)} \\
&= \frac{1}{2\pi t} \int_{\mathbb{R}} dl' \sum_{j\in\mathbb{Z}} \int_{[2\pi j, 2\pi(j+1)} d\varphi'\ \underbrace{f(\varphi'-2\pi j, l')}_{=f(\varphi', l')}\ e^{-\frac{1}{2t}((l-l')^2+(\varphi-\varphi')^2)} \\
&= \frac{1}{2\pi t} \int_{\mathbb{R}} dl' \int_{\mathbb{R}} d\varphi'\ f(\varphi', l')\ e^{-\frac{1}{2t}((l-l')^2+(\varphi-\varphi')^2)}
\end{aligned}
\tag{3.199}
$$

$$= \left(e^{\frac{t}{2}\Delta_{(\varphi,l)}}f\right)(\varphi,l),$$

where we used the invariance of the measure $dl \wedge d\varphi = (2\pi i|\xi|^2)^{-1} d\xi \wedge d\bar{\xi}$ under the substitution $\varphi' \mapsto \varphi + \pi n,\ l' \mapsto l' - i\pi n$, because $\Phi(e^{i\varphi}, l) = e^{-l+\varphi} = e^{-(l-i\pi n)+i(\varphi+\pi n)} = \Phi(e^{(-1)^n i\varphi}, l - i\pi n)$, and identified f with its periodic extension to $T^*\mathbb{R}$.

Remark III.41:
The relation $L^t_{Q^{\mathrm{B}}_t(f)}(\Phi(e^{i\varphi},l)) = (e^{\frac{t}{2}\Delta_{(\varphi,l)}}f)(\varphi,l)$ is in accordance with the covering $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \cong U(1)$ and the observation that ϑ_3 arises from a $2\pi\mathbb{Z}$-periodisation of the Euclidean heat kernel, $\frac{1}{2\pi}\vartheta_3(\frac{l}{2\pi}|\frac{it}{2\pi}) = \sum_{j\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(l-2\pi j)^2}{2t}}$. Furthermore, an analogous calculation as in (3.197) shows that the lower symbol of a product of two Berezin quantisations $Q^{\mathrm{B}}_t(f),\ Q^{\mathrm{B}}_t(f')$ satisfies

$$\begin{aligned} &L^t_{Q^{\mathrm{B}}_t(f)Q^{\mathrm{B}}_t(f')}(\Phi(e^{i\varphi},l)) \\ &= \int_{\mathbb{R}^2} \frac{d\varphi' dl'}{2\pi t}\ e^{-\frac{1}{2t}(\varphi'+il')(\varphi'-il')} f(\varphi+\sqrt{2t}\varphi', l+\sqrt{2t}l') \\ &\quad \times \int_{\mathbb{R}^2} \frac{d\varphi'' dl''}{2\pi t} e^{-\frac{1}{2t}(\varphi''+il'')(\varphi''-il'')} e^{-\frac{1}{2t}(\varphi'-il')(\varphi''+il'')} f'(\varphi+\sqrt{2t}\varphi'', l+\sqrt{2t}l''), \end{aligned} \tag{3.200}$$

which is familiar from the $\mathbb{R}^n$-case, as well.

If we were to base the Berezin quantisation on the resolution of unity (2.34),

$$L^t_{Q^{\mathrm{B}}_t(f)}(\xi,\bar{\xi}) = C_t \int_{\mathbb{C}^*} \frac{d\xi' \wedge d\bar{\xi}'}{4\pi i} |\xi'|^{-2} f(\Phi^{-1}(\xi'))\ \underbrace{\frac{|{}_t\langle \xi, j_0|\xi', j_0\rangle_t|^2}{{}_t\langle\xi, j_0|\xi, j_0\rangle_t\ {}_t\langle \xi', j_0|\xi', j_0\rangle_t}}_{=\mathrm{tr}\left(P^{j_0}_t(\xi,\bar{\xi})P^{j_0}_t(\xi',\bar{\xi}')\right)}, \tag{3.201}$$

which was already proven for $G = U(1)$, we would obtain a similar result:

$$\begin{aligned} &L^t_{Q^{\mathrm{B}}_t(f)}(\Phi(e^{i\varphi},l)) \\ &= \sum_{n\in\mathbb{Z}} \frac{e^{-\frac{\pi^2}{t}n^2} e^{-2\pi i n(\frac{l}{t}-j_0)}}{\vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})} \int_{\mathbb{R}} \frac{C_t dl'}{2\pi\ \vartheta_3(\frac{l'}{t}-j_0|\frac{i\pi}{t})} \int_{[0,2\pi)} d\varphi'\ f(\varphi',l') \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2t}((l-l'-i\pi n)^2+(\varphi-\varphi'+\pi n-2\pi j)^2)} \\ &= \frac{C_t}{2\pi}\int_{\mathbb{R}} dl'\ \frac{\vartheta_3((\frac{l}{t}-j_0)+(\frac{l'}{t}-j_0)|\frac{i2\pi}{t})}{\vartheta_3(\frac{l}{t}-j_0|\frac{i\pi}{t})\vartheta_3(\frac{l'}{t}-j_0|\frac{i\pi}{t})} \int_{\mathbb{R}} d\varphi'\ f(\varphi',l')\ e^{-\frac{1}{2t}((l-l')^2+(\varphi-\varphi')^2)}. \end{aligned} \tag{3.202}$$

There is yet another interesting way to obtain the relation between the upper and lower symbols, namely via applying the commutation relation between creation and annihilation operators,

$$X_t X_t^* = e^{2t} X_t^* X_t, \tag{3.203}$$

to any operator $A \in \mathscr{B}(L^2(U(1)))$ in (anti-)Wick-ordered form:

$$A = \sum_{m,n\in\mathbb{Z}} \left(A^W_t\right)_{mn} (X_t^*)^m X_t^n \tag{3.204}$$

$$= \sum_{m,n\in\mathbb{Z}} \underbrace{\left(A_t^W\right)_{mn} e^{-2tmn}}_{=:\left(A_t^{aW}\right)_{mn}} X_t^n (X_t^*)^m.$$

From the resolution of unity (2.30) and (3.196), we infer that the upper and lower symbol of A are given by the Laurent series:

$$\begin{aligned} f_A(\Phi^{-1}(\xi)) &= A_t^{aW}(\xi,\overline{\xi}) \\ &= \sum_{m,n\in\mathbb{Z}} \left(A_t^{aW}\right)_{mn}\ \xi^m\ \overline{\xi}^n, \\ &= \sum_{m,n\in\mathbb{Z}} e^{-2tmn}\left(A_t^{W}\right)_{mn}\ \xi^m\ \overline{\xi}^n \end{aligned} \qquad \begin{aligned} L_A^t(\xi,\overline{\xi}) &= A_t^W(\xi,\overline{\xi}) \\ &= \sum_{m,n\in\mathbb{Z}} \left(A_t^{W}\right)_{mn}\ \xi^m\ \overline{\xi}^n, \end{aligned} \tag{3.205}$$

which evidently satisfy

$$L_A^t = e^{\frac{t}{2}\Delta} f_A. \tag{3.206}$$

Unfortunately, such simple reasoning is not available for general compact Lie groups, because the commutation relations of the creation and annihilation operators do not close among themselves.

To conclude this section, we also compute the analogs of the relations (3.190) for $U(1)$-equivariant Kohn-Nirenberg[77] symbols (cp. (3.15) and (3.17)). In this case, the Weyl elements are obtained from the Weyl algebra (3.194):

$$W_t^{j_0}(\varphi,k) := \frac{t}{2\pi}\sum_{m\in\mathbb{Z}}\int_{[0,2\pi/t)} d\beta\ e^{-i(m\varphi+\beta(k+j_0))}U_t(m)V_t(\beta), \tag{3.207}$$

which satisfy the relation

$$\mathrm{tr}_{\mathfrak{H}_{j_0}}(W_t^{j_0}(\varphi,k)^* W_t^{j_0}(\varphi',k')) = 2\pi\sum_{m\in\mathbb{Z}}\delta(\varphi-\varphi'-2\pi m)\delta_{k,k'}. \tag{3.208}$$

Now, Kohn-Nirenberg quantisation and dequantisation takes the form:

$$A_{\sigma,t} := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{[0,2\pi)} d\varphi\ \sigma_t(\varphi,k)W_t^{j_0}(\varphi,k), \qquad \sigma_{A,t}(\varphi,k) := \mathrm{tr}_{\mathfrak{H}_{j_0}}(W_t^{j_0}(\varphi,k)^* A_t), \tag{3.209}$$

$$\left(A_{\sigma,t}\Psi\right)(\varphi) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{[0,2\pi)} d\varphi'\ e^{ik(\varphi-\varphi')}\sigma_t(\varphi,k)\Psi(\varphi') = \sum_{k\in\mathbb{Z}} e^{ik\varphi}\sigma_t(\varphi,k)\hat{\Psi}(k),\ \ \Psi\in C_b^\infty(\mathbb{R})\cap\mathfrak{H}_{j_0}.$$

There are natural symbol spaces associated with (3.209) (cf.[5]), which are in close analogy with those familiar from pseudo-differential operators on $\mathbb{R}^n$ ($m\in\mathbb{R}, 0\leq\delta\leq\rho\leq 1$):

$$\begin{aligned} \sigma\in S^m_{\rho,\delta}(U(1)\times\mathbb{Z})\ \ :\Leftrightarrow\ \ &\forall k\in\mathbb{Z}:\sigma(\ .\ ,k)\in C^\infty(U(1)) \\ &\&\ \forall\alpha,\beta\in\mathbb{N}_0:\forall(\varphi,k)\in U(1)\times\mathbb{Z}:\exists C_{\alpha\beta}>0: \\ &\qquad |(\partial_\varphi^\alpha\Delta_k^\beta\sigma)(\varphi,k)|\leq C_{\alpha\beta}\langle k\rangle^{m-\rho\beta+\delta\alpha}, \end{aligned} \tag{3.210}$$

$$S^{-\infty}(U(1)\times\mathbb{Z}) := \bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta}(U(1)\times\mathbb{Z}), \qquad S^{\infty}_{\rho,\delta}(U(1)\times\mathbb{Z}) := \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta}(U(1)\times\mathbb{Z}).$$

Here, $(\Delta_k f)(k) := f(k+1) - f(k)$ for $f : \mathbb{Z} \to \mathbb{C}$ is the forward difference. If we allow for $U(1)$-equivariant symbols, i.e. $\sigma(\varphi + 2\pi j, k) = e^{2\pi i j j_0}\sigma(\varphi, k)e^{-2\pi i j j_0'}$, we can encompass operators $A_\sigma : \mathfrak{H}_{j_0'} \to \mathfrak{H}_{j_0}$, as well.
Clearly, we could have invoked $U(1)$-equivariant symbols,

$$\begin{aligned} S^m_{\rho,\delta,(j_0,j_0')}(\mathbb{R}^2) &= \{\sigma \in S^m_{\rho,\delta}(\mathbb{R}^2) \mid \forall j \in \mathbb{Z} : \sigma(q+2\pi j, p) = \sigma(\varphi + 2\pi j, k) = e^{2\pi i j j_0}\sigma(q,p)e^{-2\pi i j j_0'}\} \\ &\subset S^m_{\rho,\delta}(\mathbb{R}^2), \end{aligned} \tag{3.211}$$

from the usual symbol classes instead of (3.210):

$$(A_{\sigma,t}\Psi)(q) = \frac{1}{2\pi t}\int_{\mathbb{R}} dp \int_{[0,2\pi)} dq\ e^{\frac{i}{t}p(q-q')}\sigma(q,p)\Psi(q') = \frac{1}{2\pi t}\int_{\mathbb{R}} dp\ e^{\frac{i}{t}pq}\sigma(q,p)\,\mathscr{F}_t[\Psi](p) \tag{3.212}$$

for $\Psi \in C^\infty_b(\mathbb{R}) \cap \mathfrak{H}_{j_0}$. We will comment on a similar dichotomy for almost-periodic pseudo-differential operators in the following subsection III D. In the present case the distinction is only apparent, because symbols in $S^m_{\rho,\delta}(U(1)\times\mathbb{Z})$ can be interpolated by those in $S^m_{\rho,\delta}(\mathbb{R}^2)$ (cf.[5], Corollary 4.6.13.).

To conclude the present subsection, we provide the relations between upper, lower and Kohn-Nirenberg symbols, which display a smoothing from upper to Kohn-Nirenberg to lower symbols similar to (3.190):

$$\begin{aligned} L^t_{A_{\sigma,t}}(\Phi(e^{i\varphi}, l)) &= \frac{\sqrt{2}}{2\pi\vartheta_3(\frac{l}{t} - j_0|\frac{i\pi}{t})}\sum_{k\in\mathbb{Z}}\int_{[0,2\pi)} d\varphi'\sigma_t(\varphi',k)e^{-\frac{1}{t}(l-t(k+j_0))^2} \\ &\qquad\times\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2t}((\varphi-\varphi'-2\pi j)-i(l-t(k+j_0)))^2} \\ &= \frac{\sqrt{2}}{2\pi\vartheta_3(\frac{l}{t} - j_0|\frac{i\pi}{t})}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} d\varphi'\sigma_t(\varphi',k)e^{-\frac{1}{t}(l-t(k+j_0))^2}e^{-\frac{1}{2t}((\varphi-\varphi')-i(l-t(k+j_0)))^2} \\ &= \frac{\sqrt{2}}{2\pi\vartheta_3(\frac{l}{t} - j_0|\frac{i\pi}{t})}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} d\varphi'\sigma_t(\varphi',k)e^{-\frac{1}{2t}((l-t(k+j_0))^2+(\varphi-\varphi')^2)}e^{\frac{i}{t}(\varphi-\varphi')(l-t(k+j_0))} \end{aligned} \tag{3.213}$$

$$\sigma_t(\varphi,k) = \frac{\sqrt{2}}{2\pi t}\int_{\mathbb{R}} dl' \int_{\mathbb{R}} d\varphi' f_{A_{\sigma,t}}(\varphi', l')e^{-\frac{1}{2t}((t(k+j_0)-l)^2+(\varphi-\varphi')^2)}e^{-\frac{i}{t}(\varphi-\varphi')(t(k+j_0)-l)}.$$

These formulas are obtained by a calculation completely analogous to (3.199).

D. Scaled Fourier transforms for $G = U(1)$ and an extension to $\mathbb{R}_{\mathrm{Bohr}}$

In subsection III B, we have discussed the issue of defining a ε-scaled integral transform for a compact (simply connected) Lie group G by means of the Stratonovich-Weyl-Fourier transform. While the resulting transform (3.177) on $L^2(G)$ is has its merits, when applied to systems modelled

on coadjoint orbits of G, its use is limited in the analysis of pseudo-differential operators on $C^\infty(G)$ as it is not invertible due to the fundamental discreteness inherent to representation theory of G, i.e. any irreducible representation of G is uniquely (up to isomorphism) determined by an integral dominant (real) weight $\lambda_\pi \in \overline{C} \cap I_r^*$.

In the following, first restricting to semisimple G, we will pursue the question, whether it is possible to lift this discreteness by associating a representation π_λ to any dominant weight $\lambda \in \overline{C}$ ($\overline{C} \subset \mathfrak{t}^*$ admits a natural $\mathbb{R}_+$-action as it is a convex cone). To be a bit more precise, we will consider (complex linear) representations $d\pi_\lambda$ of $\mathfrak{g}_\mathbb{C}$ resp. $U(\mathfrak{g}_\mathbb{C})$[78] instead of G to lift the integrality condition. A natural way to achieve this is to exploit the construction of irreducible representations of G by means of Verma modules (cf.[60,79,80]). Since the representation theory of $\mathfrak{g}_\mathbb{C}$ is typically formulated with respect to (infinitesimal) integral weights $\lambda \in 2\pi i I_r^* = I^*$, and their complex linear extensions to $\mathfrak{t}_\mathbb{C}^*$, instead of integral real weights, we will switch to using the former for the reminder of the section. The same applies to the roots of $\mathfrak{g}$, and the subset of positive roots R^+.

Definition III.42 (cf.[79], I.1.3.):
Given $\lambda \in \mathfrak{t}^*$, the $U(\mathfrak{g}_\mathbb{C})$-module

$$M(\lambda) := U(\mathfrak{g}_\mathbb{C}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_\lambda \tag{3.214}$$

is called the Verma module of highest weight λ. *Here,*

$$\mathfrak{b}^+ := \mathfrak{t}_\mathbb{C} \oplus \underbrace{\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}_{=:\mathfrak{n}^+} \tag{3.215}$$

is the standard Borel subalgebra *of $\mathfrak{g}_\mathbb{C}$ associated with R^+, and $\mathbb{C}_\lambda$ is the 1-dimensional $U(\mathfrak{b}^+)$-module defined by*

$$d\pi_\lambda(H+N)1 := \lambda(H)1,\ H \in \mathfrak{t}_\mathbb{C},\ N \in \mathfrak{n}^+\,. \tag{3.216}$$

We denote by $L(\lambda) := M(\lambda)/N(\lambda)$ the unique irreducible quotient module w.r.t. to the unique maximal submodule $N(\lambda) \subset M(\lambda)$ (cf.[79], Theorem I.1.2.).

Remark III.43:
The Verma module construction (3.214) also works for $G = U(1)^n$. Since $R^+ = \emptyset$ and $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \cong \mathbb{C}^n$ in this case, we have $M(\lambda) = \mathbb{C}_\lambda$, which is irreducible.
For semisimple G, the irreducible representation $L(\lambda)$ of $\mathfrak{g}_\mathbb{C}$ is finite dimensional if and only if λ is dominant integral. $M(\lambda)$ is freely generated by $U(\mathfrak{n}^-)(1 \otimes 1),\ \mathfrak{n}^- := \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$.

Unfortunately, the reason, why $L(\lambda)$ does not integrate to a (unitary) representation of G, when λ is not dominant integral, can be traced back to the fact, that there is no Hilbert space structure on $L(\lambda)$ compatible with the adjointness relations $d\pi_\lambda(E_\alpha)^* = d\pi_\lambda(E_{-\alpha}),\ \alpha \in R^+$, and $d\pi_\lambda(H_i)^* = d\pi_\lambda(H_i),\ i = 1, ..., r$, for a choice of Cartan-Weyl basis (3.153) (cf.[81]) but only a Krein space structure. This is easily seen, in the case of $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_2(\overline{C} \cap I^* \cong \mathbb{N},\ R^+ = \{2\})$:

$$\begin{aligned}(d\pi_\lambda(E_{-2})^k v_\lambda, d\pi_\lambda(E_{-2})^k v_\lambda)_\lambda &= (v_\lambda, d\pi_\lambda(E_2)^k d\pi_\lambda(E_{-2})^k v_\lambda)_\lambda \\ &= (v_\lambda, d\pi_\lambda(E_2)^{k-1}(d\pi_\lambda(E_{-2})d\pi_\lambda(E_2) + d\pi_\lambda(H_2))d\pi_\lambda(E_{-2})^{k-1} v_\lambda)_\lambda \\ &= k(\lambda(H_2) + 1 - k)(d\pi_\lambda(E_{-2})^{k-1} v_\lambda, d\pi_\lambda(E_{-2})^{k-1} v_\lambda)_\lambda\end{aligned} \tag{3.217}$$

$$= \left(\prod_{l=1}^{k} l(\lambda(H_2) + 1 - l) \right) (v_\lambda, v_\lambda)_\lambda, \;\; k \in \mathbb{N}_0$$

where we denoted the highest weight vector $1 \otimes 1 \in M(\lambda)$ by v_λ, and invoked the commutation relations (3.153). The last line in (3.217) shows that the inner product of $v_\lambda^k = d\pi_\lambda(E_{-2})^k v_\lambda, \; k \in \mathbb{N}_0$, becomes negative for certain $k > \lambda(H_2)+1$, if $\lambda(H_2) \notin \mathbb{N}$. Moreover, the modules $M(\lambda), \; \lambda(H_2) \notin \mathbb{N}$, are irreducible implying that there is no compatible Hilbert space structure. For $\lambda(H_2) \in \mathbb{N}$, we obtain the submodule $N(\lambda) = \mathrm{span}_{\mathbb{C}}\{v_\lambda^k \mid k \geq \lambda(H_2)+1\}$ of null vectors, which can be factored out. A similar argument applies to general semisimple $\mathfrak{g}_{\mathbb{C}}$ by an appeal to the $\mathfrak{sl}_2$-submodules generated by $\{E_\alpha, E_{-\alpha}, H_\alpha\}, \; \alpha \in R^+$.

In contrast, if $G = U(1)^n$ the set of roots is empty, and the standard inner product on $\mathbb{C}_\lambda \cong \mathbb{C}$ is compatible with the adjointness relations. Furthermore, the representation $\mathbb{C}_\lambda$ integrates to a (unitary) representation of $\mathbb{R}^n$:

$$\pi(x \cdot H)_\lambda 1 = e^{i \sum_{i=1}^{n} x_i \lambda(H_i)} 1, \;\; x \cdot H = \sum_{i=1}^{n} x_i H_i, \;\; x \in \mathbb{R}^n \,. \tag{3.218}$$

Thus, for $G = U(1)^n$ we may form the direct sums of Hilbert spaces

$$\mathfrak{H}_n := \bigoplus_{\{\lambda(H_i)\}_{i=1}^n \in \mathbb{R}^n} \mathbb{C}_\lambda, \;\; n \in \mathbb{N}, \qquad\qquad \mathfrak{H}_n \cong \mathfrak{H}_1^{\otimes n} \,. \tag{3.219}$$

The Hilbert space $\mathfrak{H}_1 = l^2(\mathbb{R})$ can be identified with $L^2(\mathbb{R}_{\mathrm{Bohr}})$, the L^2-space on the Bohr compactification of $\mathbb{R}$, which can be defined as the L^2-closure of $\mathrm{span}_{\mathbb{C}}\{e_\lambda : \mathbb{R} \to \mathbb{T} \mid \lambda \in \mathbb{R}\}, \; e_\lambda(x) := e^{i\lambda x}$, w.r.t. the ergodic mean (the Haar measure on $\mathbb{R}_{\mathrm{Bohr}}$)

$$(f, f')_{\mathrm{Bohr}} = \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x) \; \overline{f(x)} f'(x) := \lim_{T \to \infty} \frac{1}{2T} \int_{[-T,T]} dx \; \overline{f(x)} f'(x). \tag{3.220}$$

$L^2(\mathbb{R}_{\mathrm{Bohr}})$ is also naturally isomorphic with the space of Besicovitch almost periodic functions $B^2(\mathbb{R})$ on $\mathbb{R}$ (cf.[9,82]). In analogy with the Bloch-Floquet transform of $L^2(\mathbb{R})$, we have a direct sum (instead of direct integral) decomposition over the elementary cell $[0,1)$ (Brillouin zone) w.r.t. the Hilbert spaces $\mathfrak{H}_{j_0}$ (see (3.191)):

$$L^2(\mathbb{R}_{\mathrm{Bohr}}) \cong \bigoplus_{j_0 \in [0,1)} \mathfrak{H}_{j_0} \,. \tag{3.221}$$

Interestingly, a similar function space realization can be obtained for the simple quotients of the Verma modules of $\mathfrak{sl}_2$. Namely, we realise $L(\lambda), \; \lambda = \lambda(H_2)$, as a subspace of $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ by

$$\begin{aligned} v_\lambda(x,y) &:= e_\lambda(x), & d\pi_\lambda(H_2) &:= -i(\partial_x - \partial_y) \\ d\pi_\lambda(E_2) &:= -ie_1(x-y)\partial_y, & d\pi_\lambda(E_{-2}) &:= -ie_{-1}(x-y)\partial_x. \end{aligned} \tag{3.222}$$

Then, $L(\lambda)$ is spanned by the weight vectors:

$$v_\lambda^k(x,y) := \left(d\pi_\lambda(E_{-2})^k v_\lambda\right)(x,y) = \left(\prod_{l=1}^{k}(\lambda+1-l)\right) e_{\lambda-k}(x)e_k(y),\ k\in\mathbb{N}_0, \tag{3.223}$$
$$v_\lambda^0(x,y) := v_\lambda(x,y),$$

which satisfy $v_\lambda^k(x+2\pi m, y+2\pi n) = e^{2\pi i j_\lambda m} v_\lambda^k(x,y),\ j_\lambda = \lambda \mod 1 \in [0,1),\ m,n\in\mathbb{Z}$. Therefore, $L(\lambda)$ constitutes a (diagonal) subspace of $\mathfrak{H}_{j_\lambda}\otimes\mathfrak{H}_0 \cong \{\Psi\in L^2_{\mathrm{loc}}(\mathbb{R}^2)\mid \forall m,n\in\mathbb{Z} : \Psi(x+2\pi m, y+2\pi n) = e^{2\pi i j_\lambda m}\Psi(x,y)\} =: L^2_{(j_\lambda,0)}(\mathbb{R}^2)$. If $L(\lambda)$ is endowed with the inner product $(\ .\ ,\ .\ )_{(j_\lambda,0)}$ coming from $\mathfrak{H}_{j_\lambda}\otimes\mathfrak{H}_0$, we will have:

$$(v_\lambda^k, v_\lambda^{k'})_{(j_\lambda,0)} = \left(k!\binom{\lambda}{k}\right)^2 \delta_{k,k'},\ k,k'\in\mathbb{N}_0, \tag{3.224}$$

implying the (modified) adjointness relations

$$d\pi_\lambda(H_2)^* = d\pi_\lambda(H_2), \tag{3.225}$$
$$d\pi_\lambda(E_2)^*\left(\lambda + d\pi_\lambda(H_2)\right)v_\lambda^k = \left(\lambda - d\pi_\lambda(H_2)\right)d\pi_\lambda(E_{-2})v_\lambda^k,\ k\in\mathbb{N}_0\,.$$

Here, we consistently set $d\pi_\lambda(E_2)^* v_\lambda^\lambda = 0$ for $\lambda\in\mathbb{N}_0$. Thus, as in (3.217), we find that the tension between Hilbert space structure and adjointness relations is due to the root vectors $E_2,\ E_{-2}$, while the adjointness relation for the generator of the Cartan subalgebra H_2 is the expected one. The completion $\hat{L}(\lambda)$ of $L(\lambda)$ w.r.t. the Hilbert space structure induced by $(\ .\ ,\ .\ )_{(j_\lambda,0)}$ is given by (possibly finite) l^2-series w.r.t. the vectors $v_\lambda^k,\ k\in\mathbb{N}_0$:

$$\Psi\in\hat{L}(\lambda)\ :\Leftrightarrow\ \Psi = \sum_{k\in\mathbb{N}_0}\Psi_k\left(k!\binom{\lambda}{k}\right)^{-1} v_\lambda^k,\ \sum_{k\in\mathbb{N}_0}|\Psi_k|^2<\infty. \tag{3.226}$$

The realisation (3.222) of $L(\lambda)$ is closely related to the Schwinger representation of $\mathfrak{sl}_2$ on $\mathbb{C}[X,Y]$:

$$\begin{aligned} a_+^* &= X := e_1(x), & a_+ &= \partial_X := -ie_{-1}(x)\partial_x,\\ a_-^* &= Y := e_1(y), & a_- &= \partial_Y := -ie_{-1}(y)\partial_y,\\ E_{\pm2} &\doteq a_\pm^* a_\mp, & H_2 &\doteq a_+^*a_+ - a_-^*a_-, \end{aligned} \tag{3.227}$$

which can be readily generalised to abtritrary $\mathfrak{g}_\mathbb{C}$. Namely, given a (complex) linear, finite dimensional representation $d\pi : \mathfrak{g}_\mathbb{C}\to\mathrm{End}(V_\pi)$ and d_π copies of the CCR-algebra, e.g. creation and annihilation operators on the (bosonic) Fock space $\mathscr{F}_s(\mathbb{C}^{d_\pi}) = \bigoplus_{n=0}^\infty \mathrm{Sym}^n\,\mathbb{C}^{d_\pi}$, we define

$$S_\pi^{g_1,g_2}(X) := \sum_{m,n=1}^{d_\pi} d\pi(X)_{mn}\left(a_m^*a_n + g_1 a_m^* + g_2 a_n + g_1g_2\right),\ X\in\mathfrak{g}_\mathbb{C}, g_1,g_2\in\mathbb{C}, \tag{3.228}$$

which satisfies $[S_\pi^{g_1,g_2}(X), S^{g_1,g_2}(Y)] = S_\pi^{g_1,g_2}([X,Y]),\ X,Y\in\mathfrak{g}_\mathbb{C}$, and coincides with the Schwinger representation of $\mathfrak{sl}_2$ for $g_1 = 0 = g_2$ and $\pi = \pi_{\frac{1}{2}}$, the fundamental representation[83].

For the remainder of the subsection, we return to $G = U(1)^n$, and further restrict to $n = 1$, in

view of (3.219) and $(\mathbb{R}^n)_{\mathrm{Bohr}} \cong (\mathbb{R}_{\mathrm{Bohr}})^n$, since in this case the extension from integral weights to arbitrary weights works to full extent. Moreover, the "decompactification" from the spaces $\mathfrak{H}_{j_0}$, as representations of the $U(1)$-Weyl algebra (3.194), to the space $\mathfrak{H}_1 = l^2(\mathbb{R}) \cong B^2(\mathbb{R}) \cong L^2(\mathbb{R}_{\mathrm{Bohr}})$, which is a representation of the standard 1-particle Weyl algebra (obtained from the algebraic state $\omega_0(W(\alpha,\beta)) = \delta_{\alpha,0}$), not only lifts the problem of ε-scaling, but additionally allows to handle the square root necessary for replacing the Kohn-Nirenberg quantisation by a genuine Weyl quantisation. There are essentially two of the latter, the first is natural in view of the isomorphism $l^2(\mathbb{R}) \cong B^2(\mathbb{R})$ (cf.[9,10]), while the second is tied to the identification $l^2(\mathbb{R}) \cong L^2(\mathbb{R}_{\mathrm{Bohr}})$, and extends the one proposed by Fewster and Sahlmann (cf.[84]):

The Weyl elements are as usual (cp. (3.194)):

$$\begin{aligned} &W_\varepsilon(\alpha,\beta) := e^{\frac{i\varepsilon}{2}\alpha\beta} U_\varepsilon(\alpha) V_\varepsilon(\beta), \\ &W_\varepsilon(\alpha,\beta)^* = W_\varepsilon(-\alpha,-\beta),\ W_\varepsilon(0,0) = \mathbb{1}, \\ &W_\varepsilon(\alpha,\beta) W_\varepsilon(\gamma,\delta) = e^{-\frac{i\varepsilon}{2}(\alpha\delta-\gamma\beta)} W_\varepsilon(\alpha+\gamma,\beta+\delta),\ \alpha,\beta,\gamma,\delta \in \mathbb{R}\,. \end{aligned} \tag{3.229}$$

Let us also introduce some frequently used test function and Sobolev spaces on $\mathbb{R}_{\mathrm{Bohr}}$ and its (topological) dual group $\mathbb{R}_{\mathrm{disc}}$ ($\mathbb{R}$ with the discrete topology, cf.[9]):

$$\begin{aligned} d(\mathbb{R}) &:= \{\hat{\Psi} : \mathbb{R} \to \mathbb{C} \mid \mathrm{supp}(\hat{\Psi}) \text{ finite}\}, & \check{d}(\mathbb{R}) &:= \mathrm{span}_{\mathbb{C}}\{e_\lambda \mid \lambda \in \mathbb{R}\} \\ & & &= \mathrm{Trig}(\mathbb{R}), \\ d'(\mathbb{R}) &:= \mathbb{C}^{\mathbb{R}}, & \check{d}'(\mathbb{R}) &:= \{T = \sum_{\lambda\in\mathbb{R}} \hat{T}(\lambda) e_\lambda \mid \hat{T} \in \mathbb{C}^{\mathbb{R}}\} \\ & & &= \mathrm{Trig}'(\mathbb{R}), \\ H^s_p(\mathbb{R}_{\mathrm{Bohr}}) &:= \overline{\mathrm{Trig}(\mathbb{R})}^{||\,\cdot\,||_{(s,p)}}, & \Big|\Big|\sum_{\lambda\in\mathbb{R}} \hat{\Psi}(\lambda) e_\lambda(x)\Big|\Big|_{(s,p)} &:= \Big(\sum_{\lambda\in\mathbb{R}} (\langle\lambda\rangle^s |\hat{\Psi}(\lambda)|)^p\Big)^{\frac{1}{p}}, \\ H^s_\infty(\mathbb{R}_{\mathrm{Bohr}}) &:= \{T \in \mathrm{Trig}'(\mathbb{R}) \mid ||T||_{s,\infty} < \infty\}, & ||T||_{(s,\infty)} &:= \sup_{\lambda\in\mathbb{R}} |\langle\lambda\rangle^s \hat{T}(\lambda)| \\ H^\infty_p(\mathbb{R}_{\mathrm{Bohr}}) &:= \bigcap_{s\in\mathbb{R}} H^s_p(\mathbb{R}_{\mathrm{Bohr}}), & H^{-\infty}_p(\mathbb{R}_{\mathrm{Bohr}}) &:= \bigcup_{s\in\mathbb{R}} H^s_p(\mathbb{R}_{\mathrm{Bohr}}). \end{aligned} \tag{3.230}$$

for $s \in \mathbb{R}, p \in [0,\infty)$. The space $d(\mathbb{R})$ bears some similarities with the space of test functions $\mathscr{D}(\mathbb{R}) = C^\infty_0(\mathbb{R})$, especially we may endowed it with the strict inductive limit topology coming from

$$d(\mathbb{R}) = \bigcup_{\substack{F\subset\mathbb{R}\\|F|<\infty}} d_F(\mathbb{R}), \qquad d_F(\mathbb{R}) := \{\hat{\Psi} : \mathbb{R} \to \mathbb{C} \mid \mathrm{supp}(\hat{\Psi}) \subset F\} \cong \mathbb{C}^{|F|}, \tag{3.231}$$

$$||\hat{\Psi}||_{F,k} := \max_{\lambda\in F} |\langle\lambda\rangle^k \hat{\Psi}(\lambda)|, \quad k \in \mathbb{N}_0,$$

although the limit is uncountable in this case. There is a natural isomorphism, $\overline{\mathrm{Trig}(\mathbb{R})}^{||\,\cdot\,||_\infty} = CAP(\mathbb{R}) \cong C(\mathbb{R}_{\mathrm{Bohr}})$, between the uniformly almost periodic functions on $\mathbb{R}$ and the continuous function on $\mathbb{R}_{\mathrm{Bohr}}$, which allows to define the space of smooth functions on $\mathbb{R}_{\mathrm{Bohr}}$:

$$C^\infty(\mathbb{R}_{\mathrm{Bohr}}) = CAP^\infty(\mathbb{R}) := CAP(\mathbb{R}) \cap C^\infty_b(\mathbb{R}), \tag{3.232}$$

carrying the natural Fréchet space topology induced by $C_b^\infty(\mathbb{R})$. Furthermore, we have $H_2^0(\mathbb{R}_{\text{Bohr}}) = L^2(\mathbb{R}_{\text{Bohr}}) \cong B^2(\mathbb{R}_{\text{Bohr}})$, and $H_1^0(\mathbb{R}_{\text{Bohr}}) \subset CAP(\mathbb{R}) \subset H_2^0(\mathbb{R}_{\text{Bohr}})$, $H_1^\infty(\mathbb{R}_{\text{Bohr}}) \subset CAP^\infty(\mathbb{R}) \subset H_2^\infty(\mathbb{R}_{\text{Bohr}})$, $H_p^s(\mathbb{R}_{\text{Bohr}}) \subset H_{p'}^{s'}(\mathbb{R}_{\text{Bohr}})$, $p \leq p'$, $s' \leq s$[85]. But $H_p^\infty(\mathbb{R}_{\text{Bohr}})$ is not embedded in $CAP(\mathbb{R})$ for $p > 1$ (in contrast with the usual situation for Sobolev spaces on $\mathbb{R}$ or $\mathbb{T}$), which can be related to $\mathbb{R}_{\text{disc}}$ not being σ-finite w.r.t. the counting measure, e.g. the function,

$$\hat{\Psi}(\lambda) := \begin{cases} \frac{1}{n} & : \ \lambda = \frac{1}{n}, \ n \in \mathbb{N} \\ 0 & : \qquad \text{else} \end{cases}, \tag{3.233}$$

belongs to $\bigcap_{p>1} H_p^\infty(\mathbb{R}_{\text{Bohr}})$, since $\forall n \in \mathbb{N}: \ 2^{-|ps|} \leq \langle \frac{1}{n} \rangle^{ps} \leq 2^{|ps|}$, while it is not in $H_1^\infty(\mathbb{R}_{\text{Bohr}})$, because the harmonic series is divergent. The inner products of the various realizations of $\mathfrak{H}_1$ induce dualities between $H_p^s(\mathbb{R}_{\text{Bohr}})$, $d(\mathbb{R})$, $\text{Trig}(\mathbb{R})$ and $H_q^{-s}(\mathbb{R}_{\text{Bohr}})$, $d'(\mathbb{R})$, $\text{Trig}'(\mathbb{R})$ for $\frac{1}{p} + \frac{1}{q} = 1$, and $\hat{}$ (or $\check{}$) in (3.230) indicates that the corresponding vectors or spaces are related via the (inverse) Fourier transform on $\mathbb{R}_{\text{Bohr}}$ (or $\mathbb{R}_{\text{disc}}$):

$$\hat{\Psi}(\lambda) := (e_{-\lambda}, \Psi)_{\text{Bohr}}, \ \Psi \in H_1^0(\mathbb{R}_{\text{Bohr}}), \qquad \check{\Phi}(x) := (e_x, \Phi)_{l^2(\mathbb{R})}, \ \Phi \in l^1(\mathbb{R}). \tag{3.234}$$

The images of the spaces $H_p^s(\mathbb{R}_{\text{Bohr}})$ under the Fourier transform are denoted by $h_p^s(\mathbb{R})$. Before we introduce the aforementioned Weyl quantisations, we add a short

Remark III.44:
We might ask, whether there is a analogue $s(\mathbb{R}) \subset l^2(\mathbb{R})$ of the Schwartz space $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Since $\mathscr{S}(\mathbb{R})$ is typically defined in terms of the uniform boundedness of expression of the form $(\langle x \rangle^m \partial_x^n f)(x)$, $m, n \in \mathbb{N}_0$, we need a viable substitute for the partial derivatives as elements of $l^2(\mathbb{R})$ are never differentiable. Additionally, we expect $\check{s}(\mathbb{R}) \subset H_p^\infty(\mathbb{R}_{\text{Bohr}})$ $\forall p \in [1, \infty]$ to hold, which requires a summability condition on the elements of $s(\mathbb{R})$, because decay conditions at infinity do not suffice in view of (3.233). A natural discretised replacement of the partial derivatives, already familiar from the context of $s(\mathbb{Z}) \subset l^2(\mathbb{Z})$, is the (scaled) forward difference $(\Delta_{x,x_0} f)(x) := f(x + x_0) - f(x)$, and because of the relative nesting of the Sobolev spaces, we could propose the following definition:

$$s(\mathbb{R}) := \{\hat{\Psi} : \mathbb{R} \to \mathbb{C} \ | \ \sum_{\lambda \in \mathbb{R}} \langle \lambda \rangle^m |(\Delta_{\lambda,\lambda_0}^n \hat{\Psi})(\lambda)| < \infty\}, \ \lambda_0 \in \mathbb{R}, \ m, n \in \mathbb{N}_0\,. \tag{3.235}$$

But, as $(\Delta_{\lambda,\lambda_0}^n \hat{\Psi})(\lambda) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \hat{\Psi}(\lambda + k\lambda_0)$, it is easy to see that $\check{s}(\mathbb{R}) = H_1^\infty(\mathbb{R}_{\text{Bohr}})$ in this case. Similar observations hold if we replace the forward difference by the backward or central difference. Interestingly, there is a useful duality between $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ and $s(\mathbb{R}) \subset l^2(\mathbb{R})$ respectively $H_1^\infty(\mathbb{R}_{\text{Bohr}}) \subset L^2(\mathbb{R}_{\text{Bohr}})$, which is compatible with inner products, Fourier transforms and multiplier actions (cp.[84]). Namely, for $f \in \mathscr{S}(\mathbb{R})$ and $\hat{\Psi} \in s(\mathbb{R})$ ($\Psi \in H_1^\infty(\mathbb{R}_{\text{Bohr}})$), we have

$$\begin{aligned}
(\mathscr{F}[f], \hat{\Psi})_{l^2(\mathbb{R})} &= \sum_{\lambda \in \mathbb{R}} \overline{\mathscr{F}[f](\lambda)} \hat{\Psi}(\lambda), \qquad |(\mathscr{F}[f], \hat{\Psi})_{l^2(\mathbb{R})}| \leq ||\Psi||_{(s,1)} \sup_{\lambda \in \mathbb{R}} \langle \lambda \rangle^{-s} |\mathscr{F}[f](\lambda)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad < \infty, \\
(\mathscr{F}[f], \hat{\Psi})_{l^2(\mathbb{R})} &= (\check{\mathscr{F}[f]}, \Psi)_{\text{Bohr}} \\
&= (f, \Psi)_{L^2(\mathbb{R})} \\
&= (\mathscr{F}[f], \mathscr{F}[\Psi])_{L^2(\mathbb{R})},
\end{aligned} \tag{3.236}$$

because $\Psi \cdot f \in \mathscr{S}(\mathbb{R})$, $\mathscr{F}[f] \cdot \hat{\Psi} \in s(\mathbb{R})$.
Furthermore, the duality is compatible with the representations π_F and π_0 of the Weyl algebra on $L^2(\mathbb{R})$ and $l^2(\mathbb{R})$, respectively:

$$\begin{aligned}(\pi_F(W_\varepsilon(\alpha,\beta)^*)f,\Psi)_{L^2(\mathbb{R})} &= (\pi_F(W_\varepsilon(\beta,-\alpha)^*)\,\mathscr{F}_\varepsilon[f],\hat{\Psi}_{(\varepsilon)})_{l^2(\mathbb{R})} \\ &= (\mathscr{F}_\varepsilon[f],\pi_0(W_\varepsilon(\beta,-\alpha))\hat{\Psi}_{(\varepsilon)})_{l^2(\mathbb{R})} \\ &= (f,\pi_0(W_\varepsilon(\alpha,\beta))\Psi)_{L^2(\mathbb{R})}.\end{aligned} \tag{3.237}$$

Here, we used the ε-scaled version of (3.234): $\hat{\Psi}_{(\varepsilon)}(\lambda) = (e_{-\frac{\lambda}{\varepsilon}},\Psi)_{\mathrm{Bohr}}$ (cp. also (3.122), although we switched the sign of the Fourier exponential, as to fit with the use of left convolution kernels in the case of compact Lie groups).

The two choices of Weyl quantisation associated with $B^2(\mathbb{R})$ and $l^2(\mathbb{R})$, respectively, $L^2(\mathbb{R}_{\mathrm{Bohr}})$ resemble the dichotomy already mentioned in the previous subsection III C (see the comment below (3.210)), and arise from (so far formal) expressions:

$$(A_\sigma\Psi)(x) = \frac{1}{2\pi\varepsilon}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} dx'\ \sigma(\tfrac{1}{2}(x+x'),\lambda)e_{-\lambda}(\tfrac{x-x'}{\varepsilon})\Psi(x'),\ \ \Psi \in \mathrm{Trig}(\mathbb{R}) \text{ or } CAP^\infty(\mathbb{R}), \tag{3.238}$$

and

$$(A_\sigma\Phi)(\lambda) = \sum_{\lambda'\in\mathbb{R}}\int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x)\ \sigma(x,\tfrac{1}{2}(\lambda+\lambda'))e_{\frac{\lambda-\lambda'}{\varepsilon}}(x)\Phi(\lambda'),\ \ \Phi \in d(\mathbb{R}) \text{ or } \check{\Phi} \in H^\infty_1(\mathbb{R}_{\mathrm{Bohr}}), \tag{3.239}$$

$$(A_\sigma\Psi)(x) = \sum_{\lambda\in\mathbb{R}}\int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x')\ \sigma(\tfrac{1}{2}(x+x'),\lambda)e_{-\lambda}(\tfrac{x-x'}{\varepsilon})\Psi(x'),\ \ \Psi \in \mathrm{Trig}(\mathbb{R}) \text{ or } H^\infty_1(\mathbb{R}_{\mathrm{Bohr}}).$$

Fundamentals on the theory of operators defined by (3.238) in Kohn-Nirenberg form can be found in the works of Shubin[9,10] (cf. also[86]), which we recall to some extent, as to allow for immediate comparison with operators of the form (3.239). Following this, we will make precise the definition of the latter.
Pseudo-differential operators like (3.238) with $\sigma \in S^m_{\rho,\delta}(\mathbb{R}^2)$ (Hörmander's symbol classes) can be defined on $C^\infty_b(\mathbb{R})$, which contains $\mathrm{Trig}(\mathbb{R})$ and $CAP^\infty(\mathbb{R})$, by the usual means of oscillatory integrals (cf.[10]). In the context of almost periodic functions, the admissible symbols[87] $APS^m_{\rho,\delta}(\mathbb{R}^2) \subset S^m_{\rho,\delta}(\mathbb{R}^2)$ are adapted to preserve the property of almost periodicity, i.e.

$$\begin{aligned}\sigma \in APS^m_{\rho,\delta}(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2)\ \ :\Leftrightarrow\ \ & \mathbb{R} \ni \lambda \mapsto \sigma(\ .\ ,\lambda) \in CAP(\mathbb{R}) \text{ is continuous} \\ & \&\forall \alpha,\beta \in \mathbb{N}_0 : \forall (x,\lambda) \in \mathbb{R}^2 : \exists C_{\alpha\beta} > 0 : \\ & \qquad |(\partial^\alpha_x\partial^\beta_\lambda\sigma)(x,\lambda)| \leq C_{\alpha\beta}\langle\lambda\rangle^{m-\rho\beta+\delta\alpha},\end{aligned} \tag{3.240}$$

$$APS^{-\infty}(\mathbb{R}^2) := \bigcap_{m\in\mathbb{R}} APS^m_{\rho,\delta}(\mathbb{R}^2), \qquad APS^\infty_{\rho,\delta}(\mathbb{R}^2) := \bigcup_{m\in\mathbb{R}} APS^m_{\rho,\delta}(\mathbb{R}^2)$$

for $m \in \mathbb{R}, 0 \leq \delta \leq \rho \leq 1$

Definition III.45:

An operator A_σ on $CAP^\infty(\mathbb{R})$ defined by (3.238) *with $\sigma \in APS^m_{\rho,\delta}(\mathbb{R}^2)$ is called an almost-periodic pseudo-differential operator.*

Statements familiar form the theory of pseudo-differential operators on $\mathbb{R}^n$ about composition, adjoints, asymptotic expansions and continuity w.r.t. to $CAP^\infty(\mathbb{R})$ and the scales of Sobolev spaces $H^s_p(\mathbb{R}_{\text{Bohr}})$ remain valid, as expected, because all necessary operations on the symbols preserve almost-periodicity. An especially interesting and useful property of almost-periodic pseudo-differential operators is

Proposition III.46 (cf.[10]):

Given A_σ with $\sigma \in APS^m_{\rho,\delta}(\mathbb{R}^2)$, $\delta < \rho$, there exist the formal adjoint A^*_σ w.r.t. to $(\ ,\ )_{L^2(\mathbb{R})}$ and $(\ ,\ )_{\text{Bohr}}$, i.e.

$$(A^*_\sigma \Psi, \Phi)_{L^2(\mathbb{R})} = (\Psi, A_\sigma \Phi)_{L^2(\mathbb{R})},\ \ \Psi, \Phi \in \mathscr{S}(\mathbb{R}), \tag{3.241}$$

$$(A^*_\sigma \Psi, \Phi)_{\text{Bohr}} = (\Psi, A_\sigma \Phi)_{\text{Bohr}},\ \ \Psi \in H^\infty_p(\mathbb{R}_{\text{Bohr}}),\ \ \Phi \in H^\infty_q(\mathbb{R}_{\text{Bohr}})\ (\tfrac{1}{p} + \tfrac{1}{q} = 1). \tag{3.242}$$

The symbol of A^*_σ is $\overline{\sigma}$.

Furthermore, Shubin[9] proves the equality $||A_\sigma||_{\mathscr{B}(L^2(\mathbb{R}))} = ||A_\sigma||_{\mathscr{B}(B^2(\mathbb{R}))}$ for bounded A_σ, which entails the equality of spectra $\text{spec}_{\mathscr{B}(L^2(\mathbb{R}))}(A_\sigma) = \text{spec}_{\mathscr{B}(B^2(\mathbb{R}))}(A_\sigma)$ by the preceding proposition (this continues to hold for (hypo)elliptic A_σ). Nevertheless, the quality of the spectra w.r.t. $L^2(\mathbb{R})$ and $B^2(\mathbb{R})$ can differ significantly, e.g. the spectrum of the Laplace operator Δ, which is essentially self-adjoint on $\mathscr{S}(\mathbb{R})$ and $\text{Trig}(\mathbb{R})$, respectively, is absolutely continuous in the first and pure point in the second case.

Now, we come to the definition of pseudo-differential operators in terms of (3.239), which we will call *Bohrian pseudo-differential operators*. Let us first remark that the formulas (3.239) are closer in structure to those applied in the definition of pseudo-differential operators on compact Lie groups (see subsection III A), while the almost-periodic pseudo-differential operators heavily exploit the special relation between $\mathbb{R}$ and $\mathbb{R}_{\text{Bohr}}$. The same reasoning applies to $U(1)$-equivariant pseudo-differential operators (see subsection III C), where the special relation between $\mathbb{R}$ and $U(1)$ via the covering morphism $\mathbb{R} \to \mathbb{R}/2\pi\,\mathbb{Z} \cong U(1)$ is of avail. But, in contrast with $U(1)$-equivariant operators, where the analogues, (3.209) and (3.212), of (3.238) and (3.239) are essentially equivalent due to the sparseness of $\mathbb{Z} \subset \mathbb{R}$ and the possibility of smooth interpolation, the situation will be different in the present case.

Definition III.47:

A function $\sigma : \mathbb{R}_{\text{disc}} \times \mathbb{R}_{\text{Bohr}} \to \mathbb{C}$ is called a symbol in $S^m_{\rho,\delta}(\mathbb{R}_{\text{Bohr}} \times \mathbb{R}_{\text{disc}}) = s^m_{\rho,\delta}$, *where $m \in \mathbb{R}$, $0 \le \delta < \rho \le 1$, if $\forall \lambda \in \mathbb{R} : \sigma(\ .\ , \lambda) \in H^\infty_1(\mathbb{R}_{\text{Bohr}})$ and $\forall \alpha, \beta \in \mathbb{N}_0 : \exists C_{\alpha,\beta} > 0$ s.t.:*

$$\forall (x,\lambda) \in \mathbb{R}_{\text{Bohr}} \times \mathbb{R}_{\text{disc}} : |(\partial^\alpha_x \Delta^\beta_\lambda \sigma)(x,\lambda)| \le C_{\alpha\beta} \langle\lambda\rangle^{m - \rho\beta + \delta\alpha}. \tag{3.243}$$

We set:

$$s^{-\infty} = \bigcap_{m \in \mathbb{R}} s^m_{\rho,\delta}, \qquad s^\infty_{\rho,\delta} = \bigcup_{m \in \mathbb{R}} s^m_{\rho,\delta}. \tag{3.244}$$

A simple application of the (discrete) Leibniz formula and Peetre's inequality, $\forall r, \lambda, \lambda' \in \mathbb{R} : \langle\lambda + \lambda'\rangle^r \le 2^{|r|} \langle\lambda\rangle^{|r|} \langle\lambda'\rangle^r$, gives:

Corollary III.48:
Let $\sigma \in s^m_{\rho,\delta}$ and $\tau \in s^{m'}_{\rho',\delta'}$. Then $\forall \alpha, \beta \in \mathbb{N}_0 : \partial^\alpha_x \Delta^\beta_\lambda \sigma \in s^{m-\rho\beta+\delta\alpha}$ and $\sigma\tau \in s^{m+m'}_{\min(\rho,\rho'),\max(\delta,\delta')}$.

In view of remark III.44, the definition of symbol classes $s^m_{\rho,\delta}$ for (3.239) requires summability conditions or restrictions on the Fourier spectrum supplementing the usual decay conditions.

Definition III.49:
A symbol $\sigma \in s^m_{\rho,\delta}$ is said to have polynomially bounded spectral growth of order $\gamma, \gamma' \in \mathbb{R}$*, if* $|\mathrm{supp}(\hat{\sigma}^1(\ . \ , \lambda)) \cap K_{\lambda'}| =: N_{\lambda'}(\lambda) \leq C_{\gamma,\gamma'} \langle\lambda\rangle^\gamma \langle\lambda'\rangle^{\gamma'}$ *for* $K_{\lambda'} = [-\lambda', \lambda'] \subset \mathbb{R}, \ \lambda' \in \mathbb{R}$*, where*

$$\hat{\sigma}^1(\lambda', \lambda) := \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x) e_{\lambda'}(x) \sigma(x, \lambda). \tag{3.245}$$

Clearly, symbols of polynomially bounded spectral growth of order γ form a subspace $s^{m,(\gamma,\gamma')}_{\rho,\delta} \subset s^m_{\rho,\delta}$. Furthermore, the property to be of polynomially bounded spectral growth is, on the one hand, preserved under multiplication in the Fourier domain and, on the other hand, not preserved under convolution in the Fourier domain.

An important subclass of symbols is given by

Definition III.50:
A symbol σ in $s^{m,(\gamma,\gamma'=1)}_{\rho,\delta}$ is called $U(1)_{\lambda_0}$-equivariant, *$\lambda_0 \in \mathbb{R}$, if $\forall \lambda \in \mathbb{R} : \mathrm{supp}(\hat{\sigma}^1(\ . \ , \lambda)) \subset \mathbb{Z}^{j_0}_{\lambda_0} := \lambda_0(\mathbb{Z} + j_0)$ for $j_0 \in [0, 1)$. Similarly, $T \in \mathrm{Trig}'(\mathbb{R})$ is $U(1)_{\lambda_0}$-equivariant, if $\mathrm{supp}(\hat{T}) \subset \mathbb{Z}^{j_0}_{\lambda_0}$. The various Fourier images of the $U(1)_{\lambda_0}$-equivariant versions of the function spaces on $\mathbb{R}_{\mathrm{Bohr}}$ (see(3.230) and below) are denoted by $d(\mathbb{Z}^{j_0}_{\lambda_0}), \ d'(\mathbb{Z}^{j_0}_{\lambda_0}), \ h^s_p(\mathbb{Z}^{j_0}_{\lambda_0})$, etc.*

Next, we give meaning to (3.239) for $\sigma \in s^{m,(\gamma,\gamma')}_{\rho,\delta}, \ \Phi \in d(\mathbb{R})$ in a similar fashion as one does for (3.238) with $\sigma \in S^m_{\rho,\delta}, \ \Psi \in C^\infty_b(\mathbb{R})$. To this end, we will work primarily with the first formula in (3.239), and interpret the second formula as a mnemonic for the dual operator defined by the ε-scaled Fourier transform. On the contrary, Fewster and Sahlmann[84] construct operators by means of the second formula. It is apparent that the first formula is natural, when working with the "volume representation" in loop quantum cosmological models.

Proposition III.51:
Let $\sigma \in s^{m,(\gamma,\gamma')}_{\rho,\delta}$, then $A_\sigma : d(\mathbb{R}) \to h^\infty_1(\mathbb{R})$. If σ is $U(1)_{\lambda_0}$-equivariant and $\lambda'_0 \in \mathbb{R}$ satisfies $\frac{\lambda'_0}{\lambda_0} = \frac{p}{q} \in \mathbb{Q}$, we have $A_\sigma : h^\infty_1(\mathbb{Z}^{j'_0}_{\lambda'_0}) \to h^\infty_1(\mathbb{Z}^{j''_0}_{\lambda''_0})$ for some $\lambda''_0 \in \mathbb{R}$ and $j''_0 \in [0, 1)$ ($\varepsilon = 1$).
In general, if $\sigma \in \mathrm{Trig}'(\mathbb{R}) \otimes d'(\mathbb{R})$, A_σ defines a quadratic form Q_σ on $d(\mathbb{R})$:

$$Q_\sigma(\Phi_1, \Phi_2) := (\Phi_1, A_\sigma \Phi_2)_{l^2(\mathbb{R})}, \ \Phi_1, \Phi_2 \in d(\mathbb{R}). \tag{3.246}$$

The formal adjoint *A^*_σ, defined by*

$$(A^*_\sigma \Phi_1, \Phi_2)_{l^2(\mathbb{R})} = (\Phi_1, A_\sigma \Phi_2)_{l^2(\mathbb{R})}, \ \Phi_1, \Phi_2 \in d(\mathbb{R}), \tag{3.247}$$

has symbol $\overline{\sigma}$.

Proof:

For simplicity, we restrict to $\varepsilon = 1$, since the general case only leads to some trivial rescaling in the formula to follow. First, let $\sigma \in s^{m,\gamma}_{\rho,\delta}$ and $\Phi \in d(\mathbb{R})$, then we use $e_\lambda(x) = \langle\lambda\rangle^{-2M}(1 -$

$\Delta_x)^M e_\lambda(x),\ M \in \mathbb{N}_0$, and the translation invariance of μ_{Bohr} to regularize (3.239):

$$(A_\sigma\Phi)(\lambda) = \sum_{\lambda' \in \mathrm{supp}(\Phi)} \Phi(\lambda') \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x') \langle \lambda - \lambda' \rangle^{-2M} ((1-\Delta_{x'})^M \sigma)(x', \tfrac{1}{2}(\lambda + \lambda'))) e_{\lambda-\lambda'}(x'). \tag{3.248}$$

By assumption, i.e. $|\mathrm{supp}(\Phi)| < \infty$ and $\sigma \in s^{m,(\gamma,\gamma')}_{\rho,\delta}$, $|\mathrm{supp}(A_\sigma\Phi) \cap K_\lambda| \leq C_{\gamma''}\langle\lambda\rangle^{\gamma''}$ for some $\gamma'' \in \mathbb{R}$. Therefore, for arbitrary $s \in \mathbb{R}$ and large enough $M \in \mathbb{N}_0$

$$\begin{aligned}
&||A_\sigma\Phi||_{(s,1)} \\
&\leq \sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi)} \langle\lambda\rangle^s \sum_{\lambda' \in \mathrm{supp}(\Phi)} \langle\lambda-\lambda'\rangle^{-2M} \sum_{\alpha=0}^{M} \binom{M}{\alpha} \sup_{x' \in \mathbb{R}} |(\partial^{2\alpha}_{x'}\sigma)(x', \tfrac{1}{2}(\lambda+\lambda'))||\Phi(\lambda')| \\
&\leq \sum_{\alpha=0}^{M} \binom{M}{\alpha} C_{2\alpha 0} \sum_{\lambda' \in \mathrm{supp}(\Phi)} |\Phi(\lambda')| \sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi)} \langle\lambda\rangle^s \langle\lambda-\lambda'\rangle^{-2M} \langle\tfrac{1}{2}(\lambda+\lambda')\rangle^{m+2\delta\alpha} \\
&\leq 2^{|s|+2M} \sum_{\alpha=0}^{M} \binom{M}{\alpha} C_{2\alpha 0} 2^{2|m+2\delta\alpha|} \sum_{\lambda' \in \mathrm{supp}(\Phi)} \langle\lambda'\rangle^{2M+|m+2\delta\alpha|} |\Phi(\lambda')| \sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi)} \langle\lambda\rangle^{-2(M-\delta\alpha)+s+m} \\
&< \infty.
\end{aligned} \tag{3.249}$$

Here, we used Peetre's inequality to arrive at the next to last line, and concluded finiteness of the last sum from, using the polynomial growth bound on $|\mathrm{supp}(A_\sigma\Phi) \cap K_\lambda|$,

$$\begin{aligned}
\sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi)} \langle\lambda\rangle^{-2(M-\delta\alpha)+s+m} &\leq \sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi)} \langle\lambda\rangle^{-2M(1-\delta)+s+m} \\
&= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{\lambda \in \mathrm{supp}(A_\sigma\Phi) \cap K_m \setminus K_{m-1}} \langle\lambda\rangle^{-2M(1-\delta)+s+m},\ K_{-1} = \emptyset \\
&\leq \lim_{n\to\infty} \sum_{m=0}^{n} \langle m-1\rangle^{-2M(1-\delta)+s+m} (N_m - N_{m-1}) \\
&\leq C'_{\gamma''} \lim_{n\to\infty} \sum_{m=0}^{1} \langle m-1\rangle^{-2M(1-\delta)+s+m+\gamma''},
\end{aligned} \tag{3.250}$$

which is finite for large enough M, because $\delta < 1$. To prove the second assertion, we observe that for $\Phi \in h^\infty_1(\mathbb{Z}^{j'_0}_{\lambda'_0})$ and $U(1)_{\lambda_0}$-equivariant σ:

$$(A_\sigma\Phi)(\lambda) \neq 0 \qquad \Leftrightarrow \qquad \lambda = \underbrace{\lambda - \lambda'}_{\in \mathbb{Z}^{j_0}_{\lambda_0}} + \underbrace{\lambda'}_{\in \mathbb{Z}^{j'_0}_{\lambda'_0}} \in \mathbb{Z}^{j_0}_{\lambda_0} + \mathbb{Z}^{j'_0}_{\lambda'_0}. \tag{3.251}$$

But, $Z^{j_0}_{\lambda_0} + \mathbb{Z}^{j'_0}_{\lambda'_0} = \{(\lambda_0 m + \lambda'_0 n) + (\lambda_0 j_0 + \lambda'_0 j'_0) \mid m, n \in \mathbb{Z}\} \subset \frac{\lambda_0}{q}\mathbb{Z} + \frac{\lambda_0}{q}(qj_0 + pj'_0)$. Thus, setting $\lambda''_0 := \frac{\lambda_0}{q} = \frac{\lambda'_0}{p}$ and $j''_0 := (qj_0 + pj'_0) \mod 1$, we have $A_\sigma\Phi \in d'(\mathbb{Z}^{j''_0}_{\lambda''_0})$. Now, we regularise the

expression for $A_\sigma\Phi$ in the same way as above to show that we even have $A_\sigma\Phi \in h_1^\infty(\mathbb{Z}^{j_0''}_{\lambda_0''})$ $(s \in \mathbb{R})$:

$$\begin{aligned}
&||A_\sigma\Phi||_{(s,1)} \\
&\leq \sum_{\lambda\in\mathbb{Z}^{j_0''}_{\lambda_0''}} \langle\lambda\rangle^s \sum_{\lambda'\in\mathbb{Z}^{j_0'}_{\lambda_0'}} \langle\lambda-\lambda'\rangle^{-2M} \sum_{\alpha=0}^{M} \binom{M}{\alpha} \sup_{x'\in\mathbb{R}} |(\partial_{x'}^{2\alpha}\sigma)(x', \tfrac{1}{2}(\lambda+\lambda'))||\Phi(\lambda')| \\
&\leq 2^{|s|+2M} \sum_{\alpha=0}^{M} \binom{M}{\alpha} C_{2\alpha 0} 2^{2|m+2\delta\alpha|} \sum_{\lambda'\in\mathbb{Z}^{j_0'}_{\lambda_0'}} \langle\lambda'\rangle^{2M+|m+2\delta\alpha|}|\Phi(\lambda')| \sum_{\lambda\in\mathbb{Z}^{j_0''}_{\lambda_0''}} \langle\lambda\rangle^{-2(M-\delta\alpha)+s+m} \\
&< \infty,
\end{aligned} \tag{3.252}$$

where, again, we employed Peetre's inequality, $\Phi \in h_1^\infty(\mathbb{Z}^{j_0'}_{\lambda_0'})$ and the finiteness of the last sum for large enough M $(\delta < 1)$.
The statements concerning the quadratic form defined by general A_σ and the formal adjoint A_σ^* are obvious from the finiteness of all sums and the behaviour of the Fourier transform w.r.t. complex conjugation.

□

Remark III.52:
Taking a look at (3.251), it becomes evident that the condition $\frac{\lambda_0'}{\lambda_0} \in \mathbb{Q}$ in proposition III.51 cannot be relaxed easily, because two relatively irrational lattices may generated arbitrary dense support for $A_\sigma\Phi$ in the bounded intervals K_λ.

Definition III.53:
Two operators A_σ, A_τ with $U(1)_{\lambda_0^\sigma}$-, respectively, $U(1)_{\lambda_0^\tau}$-equivariant symbols σ, τ, defined on the (rational scales of) spaces

$$h_1^\infty(\lambda_0^\sigma) := \bigcup_{\substack{\lambda_0'\in\mathbb{R} \\ \frac{\lambda_0'}{\lambda_0^\sigma}\in\mathbb{Q}}} \bigcup_{j_0'\in[0,1)} h_1^\infty(\mathbb{Z}^{j_0'}_{\lambda_0'}), \qquad h_1^\infty(\lambda_0^\tau) := \bigcup_{\substack{\lambda_0''\in\mathbb{R} \\ \frac{\lambda_0''}{\lambda_0^\tau}\in\mathbb{Q}}} \bigcup_{j_0''\in[0,1)} h_1^\infty(\mathbb{Z}^{j_0''}_{\lambda_0''}) \tag{3.253}$$

are called relatively rational, *if $\frac{\lambda_0^\sigma}{\lambda_0^\tau} \in \mathbb{Q}$ $(\varepsilon = 1)$. Clearly, in this case $h_1^\infty(\lambda_0^\sigma) = h_1^\infty(\lambda_0^\tau)$ (other function space are defined similarly[88]).*

Corollary III.54:
Relatively rational operators generate an algebra with common domain $h_1^\infty(\lambda_0)$ for some $\lambda_0 \in \mathbb{R}$.

Corollary III.55:
Let σ be a $U(1)_{\lambda_0}$-equivariant symbol, then $A_\sigma^{(\varepsilon)}$ and $A_\sigma^{(\varepsilon')}$ are relatively rational if and only if $\frac{\varepsilon'}{\varepsilon} \in \mathbb{Q}$. Here, we made the ε-dependence of (3.239) *explicit.*

If we want to use symbolic calculus in the analysis of Bohrian pseudo-differential operators, we will need a statement concerning the asymptotic summation of symbols in $s^m_{\rho,\delta}$.

Proposition III.56:

Let $\{m_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be such that $\lim_{j\to\infty} m_j = -\infty$, $m := \max_{j\in\mathbb{N}} m_j$, and $\sigma_j \in s^{m_j}_{\rho,\delta}$ for all $j \in \mathbb{N}$. Then, there exists a symbol $\sigma \in s^m_{\rho,\delta}$, unique up to $s^{-\infty}$, such that $a \sim \sum_{j=1}^{\infty} \sigma_j$, i.e.:

$$\forall n \in \mathbb{N} : \exists k_n \in \mathbb{N} : \forall k \geq k_n : \ \sigma - \sum_{j=1}^{k} \sigma_j \in s^{m_n}_{\rho,\delta}. \tag{3.254}$$

If the symbols σ_j, $j \in \mathbb{N}$, are of polynomially bounded spectral growth of order (γ, γ') or $U(1)_{\lambda_0}$-equivariant, then so is σ.

Proof:

Using standard excision function techniques and literally repeating the argument in[5], Theorem 4.4.1, accomplishes the proof. The statement concerning polynomially bounded spectral growth and equivariance follows, because the excision functions only touch the second argument of the symbols. □

We close this section with a discussion of the differences between almost-periodic pseudo-differential operators and Bohrian pseudo-differential operators:

Our first observation is, as already mentioned above, that almost-periodic pseudo-differential operators are not equivalent to Bohrian pseudo-differential operators, at least not without extending the symbol classes for almost-periodic pseudo-differential operators to include non-smooth elements, in contrast with the analogous situation in the $U(1)$-equivariant case. This is most easily inferred from an explicit example (important in loop quantum cosmology):

$$A \ : h^s_p(\mathbb{R}) \to h^{s-1}_p(\mathbb{R}), \ s \in \mathbb{R}, \qquad (A\Phi)(\lambda) := |\lambda|\Phi(\lambda), \ \Phi \in h^{s+1}_p(\mathbb{R}), \tag{3.255}$$

which has the symbol $\sigma_A(x, \lambda) = |\lambda|$, which makes sense for (3.238) as well as (3.239). Clearly, $\sigma_A \in s^{1,(0,0)}_{1,0}$, but $\sigma \notin APS^m_{\rho,\delta}$ for any m, ρ, δ, because this require smoothness in the second argument. Since an element $\Phi \in h^s_p(\mathbb{R})$ can have support at any point $\lambda \in \mathbb{R}$, there is no smooth interpolation of σ_A (the same observation holds for the spaces $h^{\infty}_1(\lambda_0)$).

Second, since symbols of almost-periodic pseudo-differential operators form a subclass of Hörmander's symbols, it is possible to transfer much of the usual symbolic calculus to their setting. This remains partially true for Bohrian pseudo-differential operator, if we replace the symbolic calculus with a discrete version familiar from the $U(1)$-equivariant case (cf.[5]). From a conceptional point of view it is useful to introduce Fourier-Weyl elements, and the associated (de-)quantisation formulas (cp. (3.4), and below):

$$\begin{aligned}
\hat{W}_{\varepsilon}(\lambda, x) &= \sum_{\beta\in\mathbb{R}} \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(\alpha) \ e_{-\lambda}(\alpha) e_{-\beta}(x) \ W_{\varepsilon}(\alpha, \beta), \qquad (3.256)\\
\mathrm{tr}_{l^2(\mathbb{R})}(\hat{W}_{\varepsilon}(\lambda, x)\hat{W}_{\varepsilon}(\lambda', x')) &= \delta_{\lambda,\lambda'}\delta_{\mathrm{Bohr}}(x - x'),\\
A_{\sigma} &= \sum_{\lambda\in\mathbb{R}} \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x) \ \sigma(x, \lambda) \ \hat{W}_{\varepsilon}(\lambda, x),\\
\sigma_A(x, \lambda) &= \mathrm{tr}_{l^2(\mathbb{R})}(\hat{W}_{\varepsilon}(\lambda, x)A),
\end{aligned}$$

which are to interpreted in a distributional sense with $\delta_{\mathrm{Bohr}} = \sum_{\lambda\in\mathbb{R}} e_{-\lambda} \in H^0_{\infty}(\mathbb{R}_{\mathrm{Bohr}})$. These formulas can be used to derive the product formula for symbols, $\rho = \sigma \star_{\varepsilon} \tau$, corresponding to the

operator product $A_\rho = A_\sigma A_\tau$ (if defined):

$$\begin{aligned}\rho(x,\lambda) &= \sum_{\lambda'\in\mathbb{R}} \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x') e_{-\lambda'}(x) e_\lambda(x') \\ &\quad \times \sum_{\lambda''\in\mathbb{R}} \int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x'') \hat{\sigma}(\lambda'', x'') \hat{\tau}(\lambda'-\lambda'', x'-x'') e_{-\frac{\varepsilon}{2}\lambda''}(x'-x'') e_{\frac{\varepsilon}{2}(\lambda'-\lambda'')}(x'') \\ &= \sum_{\lambda'\in\mathbb{R}} \sum_{\lambda''\in\mathbb{R}} e_{-\lambda'}(x) e_{-\lambda''}(x)\ \hat{\sigma}^1(\lambda'', \lambda + \tfrac{\varepsilon}{2}\lambda') \hat{\tau}^1(\lambda', \lambda - \tfrac{\varepsilon}{2}\lambda''), \\ \hat{\rho}^1(\lambda',\lambda) &= \sum_{\lambda''\in\mathbb{R}} \hat{\sigma}^1(\lambda'', \lambda + \tfrac{\varepsilon}{2}(\lambda'-\lambda'')) \hat{\tau}^1(\lambda'-\lambda'', \lambda - \tfrac{\varepsilon}{2}\lambda''),\end{aligned} \tag{3.257}$$

which is completely analogous with the formula for the standard Moyal product (3.7). For $\sigma, \tau \in \mathrm{Trig}(\mathbb{R}) \otimes d(\mathbb{R})$ the expression (3.257) is convergent, but in general, e.g. for A_σ, A_τ relatively rational, it has to be interpreted in an oscillatory sense, i.e. it should be regularised in the way we used to define A_σ.

Remark III.57:
The formula (3.257) shows that the composition $A_\rho = A_\sigma A_\tau$ of relatively rational operators A_σ, A_τ is also relatively rational to A_σ, A_τ.

Finally, we look into possible asymptotic expansions of (3.257). Let us first assume that σ and τ are smooth in the second argument, and belong to Hörmander's symbol classes. Then, we apply a Taylor expansion of the product of $\hat{\sigma}^1$ and $\hat{\tau}^1$ in (3.257):

$$\begin{aligned}\rho(x,\lambda) &= \sum_{\lambda'\in\mathbb{R}} e_{-\lambda'}(x) \sum_{\lambda''\in\mathbb{R}} e_{-\lambda''}(x) \\ &\quad \times \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{-i\varepsilon}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left((\widehat{\partial_x^k \partial_\lambda^{n-k}\sigma}^1)(\lambda'',\lambda) (\widehat{\partial_x^{n-k}\partial_\lambda^k \tau}^1)(\lambda',\lambda) \right) \\ &\quad + \left(\frac{-i\varepsilon}{2}\right)^{N+1} R^{(N)}_{\sigma,\tau}(x,\lambda), \\ &= \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{-i\varepsilon}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left((\partial_x^k \partial_\lambda^{n-k}\sigma)(\lambda'',\lambda) (\partial_x^{n-k}\partial_\lambda^k \tau)(\lambda',\lambda) \right) \\ &\quad + \left(\frac{-i\varepsilon}{2}\right)^{N+1} R^{(N)}_{\sigma,\tau}(x,\lambda), \\ R^{(N)}_{\sigma,\tau}(x,\lambda) &= \frac{1}{N!} \sum_{\lambda'\in\mathbb{R}} e_{-\lambda'}(x) \sum_{\lambda''\in\mathbb{R}} e_{-\lambda''}(x) \sum_{k=0}^{N+1} \binom{N+1}{k} (-1)^{N+1-k} \\ &\quad \times \int_0^1 dt (1-t)^N (\widehat{\partial_x^k \partial_\lambda^{n-k}\sigma}^1)(\lambda'', \lambda + \tfrac{t\varepsilon}{2}\lambda') (\widehat{\partial_x^{n-k}\partial_\lambda^k \tau}^1)(\lambda', \lambda - \tfrac{t\varepsilon}{2}\lambda''),\end{aligned} \tag{3.258}$$

which resembles the well-know formulas from the $\mathbb{R}^n$-case, besides the fact that we have to deal with the Fourier series on $\mathbb{R}_{\mathrm{disc}}$ instead of the Fourier transform, which leads to different convergence properties (see above). If we do not want to impose smoothness of the symbols and work with the

classes $s^m_{\rho,\delta}$ (see definition III.47), we will have to replace the Taylor expansion by some non-smooth analogue. A least, in the case of relatively rational operators, this is achieved by the *discrete Taylor expansion* or *Newton series* (cf.[5], Theorem 3.3.21), because this essentially reduces the situation to the $U(1)$-equivariant case. For $\Phi : \mathbb{Z}^n \to \mathbb{C}$, we have (in multi-index notation):

$$\begin{aligned}\Phi(\lambda+\lambda') &= \sum_{\substack{\alpha\in\mathbb{N}_0\\|\alpha|\le N}} \frac{1}{\alpha!}\lambda'^{(\alpha)}(\Delta^\alpha_\lambda\Phi)(\lambda) + r^N_\Phi(\lambda,\lambda'), \quad \lambda'^{(\alpha)} = \prod_{i=1}^n \lambda_i'^{(\alpha_i)},\ \lambda_i'^{(\alpha_i)} := \lambda'\cdot\ldots\cdot(\lambda'-\alpha_i+1),\\ &= \sum_{\substack{\alpha\in\mathbb{N}_0\\|\alpha|\le N}} \binom{\lambda'}{\alpha}(\Delta^\alpha_\lambda\Phi)(\lambda) + r^N_\Phi(\lambda,\lambda'), \qquad\qquad = \alpha!\binom{\lambda'}{\alpha}\end{aligned} \tag{3.259}$$

where the remainder $r^{(N)}_\Phi(\lambda,\lambda')$ satisfies:

$$\begin{aligned}|(\Delta^\beta_\lambda r^{(N)}_\Phi)(\lambda,\lambda')| &\le \max_{\substack{|\alpha|=N+1\\ \lambda''\in Q(\lambda')}} |\lambda'^{(\alpha)}(\Delta^{\alpha+\beta}_\lambda\Phi)(\lambda+\lambda'')|,\\ Q(\lambda') &:= \{\lambda''\in\mathbb{Z}^n \ \mid |\lambda''_i|\le|\lambda'_i|,\ i=1,\ldots,n\}.\end{aligned} \tag{3.260}$$

Therefore, if A_σ and A_τ are relatively rational operators with $\sigma\in s^{m_\sigma}_{\rho,\delta}$ and $\sigma\in s^{m_\tau}_{\rho,\delta}$, their composition $A_\rho = A_\sigma A_\tau$ is defined, and we find:

$$\begin{aligned}\rho(x,\lambda) &= \sum_{\lambda'\in\mathbb{Z}^{j_0^\tau}_{\lambda_0^\tau}}\ \sum_{\lambda''\in\mathbb{Z}^{j_0^\sigma}_{\lambda_0^\sigma}} e_{-\lambda'}(x)e_{-\lambda''}(x)\ \hat\sigma^1(\lambda'',\lambda+\tfrac{\varepsilon}{2}\lambda')\hat\tau^1(\lambda',\lambda-\tfrac{\varepsilon}{2}\lambda'')\\ &= e_{-(\lambda_0^\sigma j_0^\sigma+\lambda_0^\tau j_0^\tau)}(x)\sum_{m_\tau\in\mathbb{Z}}\sum_{m_\sigma\in\mathbb{Z}} e_{-(\lambda_0^\sigma m_\sigma+\lambda_0^\tau m_\tau)}(x)\ \hat\sigma^1(\lambda_0^\sigma(m_\sigma+j_0^\sigma),\lambda+\tfrac{\varepsilon}{2}\lambda_0^\tau(m_\tau+j_0^\tau))\\ &\qquad\qquad\times\hat\tau^1(\lambda_0^\tau(m_\tau+j_0^\tau),\lambda-\tfrac{\varepsilon}{2}\lambda_0^\sigma(m_\sigma+j_0^\sigma))\\ &= e_{-(\lambda_0^\sigma j_0^\sigma+\lambda_0^\tau j_0^\tau)}(x)\sum_{m_\tau\in\mathbb{Z}}\sum_{m_\sigma\in\mathbb{Z}} e_{(\lambda_0^\sigma m_\sigma-\lambda_0^\tau m_\tau)}(x)\\ &\quad\times\sum_{n=0}^N\frac{1}{n!}\sum_{k=0}^n\binom{n}{k} m_\tau^{(k)}\left(\Delta^k_{\lambda,\frac{\varepsilon}{2}\lambda_0^\tau}\hat\sigma^1\right)(\lambda_0^\sigma(-m_\sigma+j_0^\sigma),\lambda+\tfrac{\varepsilon}{2}\lambda_0^\tau j_0^\tau)\\ &\qquad\qquad\times m_\sigma^{(n-k)}\left(\Delta^{n-k}_{\lambda,\frac{\varepsilon}{2}\lambda_0^\sigma}\hat\tau^1\right)(\lambda_0^\tau(m_\tau+j_0^\tau),\lambda-\tfrac{\varepsilon}{2}\lambda_0^\sigma j_0^\sigma)\\ &\quad+r^{(N)}_{\sigma,\tau}(x,\lambda;\tfrac{\varepsilon}{2})\\ &= \sum_{n=0}^N\frac{1}{n!}\sum_{k=0}^n\binom{n}{k}\left(\left(\overline{D}^{j_0^\sigma}_{x,\lambda_0^\sigma}\right)^{(n-k)}\Delta^k_{\lambda,\frac{\varepsilon}{2}\lambda_0^\tau}\sigma\right)(x,\lambda+\tfrac{\varepsilon}{2}\lambda_0^\tau j_0^\tau)\\ &\qquad\qquad\times\left(\left(D^{j_0^\tau}_{x,\lambda_0^\tau}\right)^{(k)}\Delta^{n-k}_{\lambda,\frac{\varepsilon}{2}\lambda_0^\sigma}\tau\right)(x,\lambda-\tfrac{\varepsilon}{2}\lambda_0^\sigma j_0^\sigma)\\ &\quad+r^{(N)}_{\sigma,\tau}(x,\lambda;\tfrac{\varepsilon}{2}),\end{aligned} \tag{3.261}$$

where $D^{j_0}_{x,\lambda_0} := -\frac{i}{\lambda_0}\partial_x - j_0$, and we employed the scaled forward difference $(\Delta_{\lambda,\lambda_0}f)(\lambda) = f(\lambda+\lambda_0)-f(\lambda)$. To arrive at the last line, we applied the inverse Fourier transform (the series are in

$h_1^\infty(\mathbb{R})$) together with the identity:

$$\lambda^{(k)}\hat{\Psi}(\lambda) = \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{Bohr}}(x)\ e_\lambda(x)(i\partial_x)^{(k)}\Psi(x),\ \Psi \in H_1^\infty(\mathbb{R}_{\text{Bohr}}). \tag{3.262}$$

The formula (3.261) constitutes a discrete analogue of the usual asymptotic Weyl product formula. Although, the expansion is not manifestly given in orders of ε, it still has the crucial property that the contribution at order n belong to the symbol classes $s_{\rho,\delta}^{m_\sigma+m_\tau-n(\rho-\delta)}$ leading to contributions by strictly smaller operators (assuming a suitable version of the Calderón-Vaillancourt theorem holds, cf.[10,11,89]) with every increase in the order of the expansion for $\delta < \rho$ (use corollary III.48). Finally, we would like to conclude that (3.261) qualifies as an asymptotic expansion, which would hold true, if we were to show that $r_{\sigma,\tau}^{(N)} \in s_{\rho,\delta}^{m_\sigma+m_\tau-(N+1)(\rho-\delta)}$. In view of the positive results of Ruzhansky and Turunen for the $U(1)$-Kohn-Nirenberg calculus (cf.[5], Theorem 4.7.10), we fully expect this to be the case.

A simple boundedness theorem of Sobolev type for $U(1)_{\lambda_0}$-equivariant operators A_σ with $\sigma \in s_{0,0}^m$ (this encompasses the important case $\sigma \in s_{\rho,0}^m \subset s_{0,0}^m$) can be proved by means of the (discrete) Young's inequality (cf.[5], where a similar reasoning is applied to $U(1)$-Kohn-Nirenberg operators, Proposition 4.2.3):

Lemma III.58:
Given a function $h : \mathbb{Z}_{\lambda_0}^{j_0} \times \mathbb{Z}_{\lambda_0'}^{j_0'} \to \mathbb{C}$, s.t.

$$C_1 := \sup_{\lambda \in \mathbb{Z}_{\lambda_0}^{j_0}} \sum_{\lambda' \in \mathbb{Z}_{\lambda_0'}^{j_0'}} |h(\lambda,\lambda')| < \infty, \qquad C_2 := \sup_{\lambda' \in \mathbb{Z}_{\lambda_0'}^{j_0'}} \sum_{\lambda \in \mathbb{Z}_{\lambda_0}^{j_0}} |h(\lambda,\lambda')| < \infty, \tag{3.263}$$

we may define $\left(K_h\Phi\right)(\lambda) := \sum_{\lambda' \in \mathbb{Z}_{\lambda_0'}^{j_0'}} h(\lambda,\lambda')\Phi(\lambda')$ for all $\Phi \in h_p^0(\mathbb{Z}_{\lambda_0'}^{j_0'})$, and have:

$$||K_h\Phi||_{(0,p)} \leq C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} ||\Phi||_{(0,p)}, \quad 1 \leq p,q \leq \infty,\ \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{3.264}$$

Proof:
The results is a simple repetition of the argument presented in[5] for scales affine $\mathbb{Z}$-lattices. □

Another useful result presented in[5], which generalises to our case, is

Lemma III.59 (cp.[5], Lemma 4.2.1):
Given $\sigma \in s_{\rho,\delta}^m$, its Fourier transform w.r.t. the first variable satisfies:

$$\forall r \in \mathbb{R}_{\geq 0},\ \beta \in \mathbb{N}_0 :\ \left|\left(\Delta_\lambda^\beta \hat{\sigma}^1\right)(\lambda',\lambda)\right| \leq C_{r,\beta}\langle\lambda'\rangle^{-r}\langle\lambda\rangle^{m-\rho\beta+\delta r}. \tag{3.265}$$

Proof:

In analogy with the usual regularisation arguments, we have for $M \in \mathbb{N}_0$

$$\begin{aligned}
\left|\left(\Delta_\lambda^\beta \hat{\sigma}^1\right)(\lambda',\lambda)\right| &= \left|\int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x)\ e_{\lambda'}(x)\sigma(x,\lambda)\right| \\
&= \left|\int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x)\ \langle\lambda'\rangle^{-2M}\left((1-\partial_x^2)^M e_{\lambda'}(x)\right)\sigma(x,\lambda)\right| \\
&= \left|\int_{\mathbb{R}_{\mathrm{Bohr}}} d\mu_{\mathrm{Bohr}}(x)\ \langle\lambda'\rangle^{-2M} e_{\lambda'}(x)(1-\partial_x^2)^M\sigma(x,\lambda)\right| \\
&\leq \langle\lambda'\rangle^{-2M}\sum_{\alpha=0}^{M}\binom{M}{\alpha}C_{2\alpha\beta}\langle\lambda\rangle^{m-\rho\beta+\delta 2\alpha} \\
&\leq \underbrace{\left(\sum_{\alpha=0}^{M}\binom{M}{\alpha}C_{2\alpha\beta}\right)}_{=:C_{2M,\beta}}\langle\lambda'\rangle^{-2M}\langle\lambda\rangle^{m-\rho\beta+\delta 2M}.
\end{aligned} \tag{3.266}$$

The boundary term arising from the partial integrations vanishes, because $\forall\lambda\in\mathbb{R}:\sigma(\ .\ ,\lambda)\in C_b^\infty(\mathbb{R})$. The result follows for $M=\frac{p}{q}\in\mathbb{Q}_{\geq 0}$ from

$$\begin{aligned}
\left|\left(\Delta_\lambda^\beta \hat{\sigma}^1\right)(\lambda',\lambda)\right| &= \left(\left|\left(\Delta_\lambda^\beta \hat{\sigma}^1\right)(\lambda',\lambda)\right|^{2q}\right)^{\frac{1}{2q}} \\
&\leq \left(C_{2p,\beta}\langle\lambda'\rangle^{-2p}\langle\lambda\rangle^{m-\rho\beta+\delta 2p}\right)^{\frac{1}{2q}}\left(C_{0,\beta}\langle\lambda\rangle^{m-\rho\beta}\right)^{\frac{2q-1}{2q}} \\
&= C_{\frac{p}{q},\beta}\langle\lambda'\rangle^{-\frac{p}{q}}\langle\lambda\rangle^{m-\rho\beta+\delta\frac{p}{q}},
\end{aligned} \tag{3.267}$$

and for $M=r\in\mathbb{R}_{\geq 0}$ by continuity. □

Now, a Calderón-Vaillancourt type result can be proved:

Theorem III.60:

A $U(1)_{\lambda_0}$-equivariant operator $A_\sigma : h_1^\infty(\mathbb{Z}^{j'_0}_{\lambda'_0}) \to h_1^\infty(\mathbb{Z}^{j''_0}_{\lambda''_0})$ as in proposition III.51 with $\sigma\in s^m_{\rho,\delta}$ extends uniquely to bounded operator from $h_p^s(\mathbb{Z}^{j'_0}_{\lambda'_0})$ to $h_p^{s-t}(\mathbb{Z}^{j''_0}_{\lambda''_0})$ for $p\in[1,\infty)$ and any $s,t\in\mathbb{R}$, s.t.

$$\exists r\in\mathbb{R}_{\geq 0}:\delta r\leq t-m \quad \& \quad (1-\delta)r > (|m|-1+|t|+|s-t|). \tag{3.268}$$

In particular, this justifies to call operators with $\sigma\in s^{-\infty}$ (infinitely) smoothing. *Moreover, if $\delta=0$, we can choose $t=m$.*

Proof:

For any $t\in\mathbb{R}$, we estimate by Peetre's inequality:

$$||A_\sigma\Phi||^p_{(s-t,p)} = \sum_{\lambda''\in\mathbb{Z}^{j''_0}_{\lambda''_0}}\langle\lambda''\rangle^{p(s-t)}\left|\sum_{\lambda'\in\mathbb{Z}^{j'_0}_{\lambda'_0}}\hat{\sigma}^1(\lambda''-\lambda',\tfrac{1}{2}(\lambda''+\lambda'))\Phi(\lambda')\right|^p \tag{3.269}$$

$$\leq 2^{p|s-t|} \sum_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \left| \sum_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \langle \lambda'' - \lambda' \rangle^{|s-t|} \langle \lambda' \rangle^{-t} \hat{\sigma}^1(\lambda'' - \lambda', \tfrac{1}{2}(\lambda'' + \lambda')) \langle \lambda' \rangle^s \Phi(\lambda') \right|^p .$$

Now, setting $h_\sigma(\lambda'', \lambda') := \langle \lambda'' - \lambda' \rangle^{|s-t|} \langle \lambda' \rangle^{-t} \hat{\sigma}^1(\lambda'' - \lambda', \frac{1}{2}(\lambda'' + \lambda'))$, we would like to employ Young's inequality (see lemma III.58), which will be possible since for any $r \in \mathbb{R}_{\geq 0}$:

$$\begin{aligned}
C_1^\sigma &:= \sup_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \sum_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} |h_\sigma(\lambda'', \lambda')| \leq C_{r,0} \sup_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \sum_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \langle \lambda'' - \lambda' \rangle^{|s-t|} \langle \lambda'' \rangle^{-t} \langle \lambda'' - \lambda' \rangle^{-r} \langle \tfrac{1}{2}(\lambda'' + \lambda') \rangle^{m+\delta r} \\
&\leq 2^{3|m+\delta r|+|t|} C_{r,0} \sup_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \sum_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \langle \lambda'' - \lambda' \rangle^{-r(1-\delta)+|m|+|t|+|s-t|} \langle \lambda'' \rangle^{-t+m+\delta r} \qquad (3.270) \\
&\leq 2^{3|m+\delta r|+|t|} C_{r,0} \Big(\sup_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \langle \lambda'' \rangle^{-t+m+\delta r} \Big) \sum_{\lambda' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \langle \lambda' \rangle^{-r(1-\delta)+|m|+|t|+|s-t|},
\end{aligned}$$

$$\begin{aligned}
C_2^\sigma &:= \sup_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \sum_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} |h_\sigma(\lambda'', \lambda')| \leq C_{r,0} \sup_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \sum_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \langle \lambda'' - \lambda' \rangle^{|s-t|} \langle \lambda'' \rangle^{-t} \langle \lambda'' - \lambda' \rangle^{-r} \langle \tfrac{1}{2}(\lambda'' + \lambda') \rangle^{m+\delta r} \\
&\leq 2^{3|m+\delta r|} C_{r,0} \sup_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \sum_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \langle \lambda'' - \lambda' \rangle^{-r(1-\delta)+|m|+|s-t|} \langle \lambda' \rangle^{-t+m+\delta r} \\
&\leq 2^{3|m+\delta r|} C_{r,0} \Big(\sup_{\lambda' \in \mathbb{Z}^{j_0'}_{\lambda_0'}} \langle \lambda' \rangle^{-t+m+\delta r} \Big) \sum_{\lambda'' \in \mathbb{Z}^{j_0''}_{\lambda_0''}} \langle \lambda'' \rangle^{-r(1-\delta)+|m|+|s-t|},
\end{aligned}$$

where we used the estimate on $\hat{\sigma}^1$ from lemma III.59 and, repeatedly, Peetre's inequality. In each case, the last line follows from the rationality of $\frac{\lambda_0'}{\lambda_0''}$. Thus, if (3.268) holds, we will have $C_1^\sigma < \infty$ and $C_2^\sigma < \infty$ $(0 \leq \delta < 1)$, from which we conclude by Young's inequality:

$$||A_\sigma \Phi||^p_{(s-m,p)} \leq 2^{p|s-m|} ||K_{h_\sigma}(\langle\ .\ \rangle^s \Phi)||^p_{0,p} \leq 2^{p|s-m|} C_1^\sigma (C_2^\sigma)^{\frac{p}{q}} ||\Phi||^p_{s,p}. \qquad (3.271)$$

Finally, because $h_1^\infty(\mathbb{R}) \subset h_p^s(\mathbb{R})$ is dense for $s \in \mathbb{R},\ p \in [1, \infty)$, the existence of a unique bounded extension of A_σ follows. The remaining statements are obvious. □

IV. CONCLUSIONS AND PERSPECTIVES

In the main part of this article, we have introduced and analysed a Weyl quantisation for loop quantum gravity-type models aiming at the implementation of the program space adiabatic perturbation theory in the latter. As we have argued, a complete implementation requires the Weyl quantisation to be scalable with the quantisation parameter ε (also called adiabatic parameter) for a perturbation theory in orders of ε to be possible. Unfortunately, due to the fact, that models à la loop quantum gravity are constructed via projective limits of function algebras living on the co-

tangent bundles of compact Lie groups (in the phase space approach), there results a fundamental asymmetry in the treatment of configuration and momentum space degrees of freedom in the quantum theory. This asymmetry entails the scalability (with ε) of the local Weyl quantisation w.r.t. to Ashtekar-Isham-Lewandowski representation, if and only if the adiabatic parameter ε is associated with the momentum variables: An effect that its easily understood from the observation that the momentum space degrees of freedom are modelled by the co-tangent space directions, which possess a (local) vector space structure. In contrast, the configuration space degrees of freedom, which are modelled by the compact Lie group underlying the co-tangent bundle, do not admit suitable, i.e. compatible with the (local) commutation relations defining the transformation group algebra, ε-scale transformations, because the existence of the latter would require exponential map to be a homomorphism of Lie groups (existence of arbitrary (real) powers of group element in a homomorphic way). In the global Weyl quantisation, where the co-tangent bundle spaces are replaced by the representation theoretical dual of the Lie group, the problem of scalability also shows up for the momentum space degrees of freedom (unitary equivalence classes of irreducible representations), and manifest itself in the rigid structure of the lattice of integral highest weights. Since the representation theoretical dual and the group itself are in a one-to-one correspondence (Tannaka-Krein duality[19], Doplicher-Roberts theorem[90]), the question of scalability in the local and global setting can be unified under the theme of ε-scalable Fourier transforms, which we discussed in subsection III B & III D).

One reason, why we are not satisfied with the local Weyl quantisation, and its scalability w.r.t. to the momentum variables, is exemplified by toy models of loop quantum cosmology-type. In those, we find that it is quite natural for the adiabatic parameter ε to be associated with the configuration space degrees of freedom (holonomies of $U(1)$): A feature that may well persist in full loop quantum gravity-type models.

Another reason is implicit in the construction of the (local) calculus Paley-Wiener-Schwartz symbols, which requires an analytic momentum dependence of the quantisable functions, because, after a fibre-wise Fourier transform from the co-tangent bundle to the tangent bundle, the dual distribution is required to be of compact support. Thus, again the non-global nature of the exponential map of a compact Lie group is of some disadvantage, as it severely restricts (analyticity!) the class of quantisable function, and therefore the class of dequantisable operators.

Such analyticity condition are not necessary in the global calculus, but the problems due to compactness show up in the way mentioned before.

An almost satisfactory solution to the difficulties introduced by the compactness of the (truncated) configuration spaces, can, up to this point, only be achieved in $U(1)^n$-models, where its possible to proceed from $U(1)$ to the still compact, but representation theoretically less rigid, Bohr compactification of the reals $\mathbb{R}_{\mathrm{Bohr}}$. Regrettably, this appears not to be a viable strategy in non-Abelian models[9].

Since we also discussed the use of coherent state/Berezin quantisations for compact Lie groups (section II and subsection III C) in our research on the implementability of the ideas of Born and Oppenheimer in loop quantum gravity-type models, it should be once more pointed out, that, unless a suitable $\star$-product is constructed from these quantisations, the existence of which is highly questionable for non-compact (truncated) phase spaces, it seems to be very difficult to construct a systematic perturbation theory. Thus, while the use of coherent states is conceptually tempting, it seems to be technically and computationally disfavoured.

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[17]Recall, that the defining representation is the unique, up to equivalence, irreducible representation of $\mathscr{K}(L^2(G))$ (cf.[7], Theorem I.2.2.6.).
[18]In view of the later sections of the article, we mention that the complexifications of $U(1)$ and $SU(2)$ are, respectively, $\mathbb{C}^*$ and $SL_2(\mathbb{C})$.
[19]T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, edited by S. Axler, F. W. Gehring, and P. R. Halmos, Graduate Texts in Mathematics, Vol. 98 (Springer, 1985).
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[34]J. C. Varilly and J. M. Gracia Bondía, Annals of Physics **190**, 107 (1989).
[35]Strictly speaking, we suppressed the "classicality parameter" ε. We will comment on this issue below.
[36]We incorporate a minus sign in the definition of the group translations, $U_g = L^*_{g^{-1}}$, because we intend to work with right invariant vector fields on G while maintaining a left action.
[37]L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vol.: 3.: Pseudo-Differential Operators*, Grundlehren der mathematischen Wissenschaften, Vol. 274 (Springer, 1985).

[38] We endow $C^\infty(G) = \mathscr{D}(G)$ and $\mathscr{D}'(G)$ with their usual nuclear topologies. $\hat{\otimes}$ denotes the complete tensor product of nuclear locally convex spaces. Clearly, the adjective "nuclear" is a misnomer for "nucular", but the former is by now well established.

[39] Note, that we sometimes abuse notation and explicitly use a representative of an isomorphism class $\pi \in \hat{G}$, but the appearance of traces on the representation space in the right places makes all formulae independent of the representative.

[40] H. Amann, Unpublished manuscript (2003).

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[42] For $G = U(1), SU(2)$ we can choose the maximal neighbourhood $V = G \setminus \{-1\}$.

[43] The subscript PW stands for "Paley-Wiener", because of the Paley-Wiener-Schwartz theorem (cf.[44]), which characterises the image of $\mathscr{D}(\mathbb{R}^n)$ (and even $\mathscr{E}'(\mathbb{R}^n)$) under the Fourier transform.

[44] L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vol.: 1.: Distribution Theory and Fourier Analysis*, Grundlehren der mathematischen Wissenschaften, Vol. 256 (Springer, 1983).

[45] Since $C^\infty(G)$ is a Fréchet space in its natural topology, there is no need to distinguish between weak and strong holomorphicity (cf.[91], Theorem 3.31).

[46] A. P. Robertson and W. Robertson, *Topological Vector Spaces*, edited by F. Smithies and J. A. Todd, Cambridge Texts in Mathematics and Mathematical Physics No. 53 (Cambridge University Press, 1964).

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[54] It is even onto, if we restrict to integral coadjoint orbits.

[55] N. Wildberger, Journal of the Australian Mathematical Society (Series A) **56**, 64 (1994).

[56] For complete systems of coherent states, the upper symbol still exist for a (strongly) dense set of operators in $\mathscr{B}(\mathfrak{H})$ in the case of infinite dimensional Hilbert spaces $\mathfrak{H}$, and it is still dual to the lower symbol (cf.[53]).

[57] We have chosen K_π to be the inverse of the operator defined in[8], because in this way K_π is a smoothing operator, as we will see in the next subsection III C.

[58] The ambiguity in taking a square root of K_π is reflected by the fact that S_π is in general not irreducible as representation V_ρ of G.

[59] $\mathrm{im}(Q^{\mathrm{SW}}_\varepsilon)$ is closed.

[60] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 9 (Springer, 1972).

[61] Here, $< \ .\ ,\ .\ >_{\mathfrak{g}}$ extends to a non-degenerate bilinear form on $\mathfrak{g}_{\mathbb{C}}$.

[62] A. Erdélyi, *Asymptotic expansions*, 3 (Courier Dover Publications, 1956).

[63] This can be seen from the fact that $\{H_\alpha E_\alpha, E_{-\alpha}\}$ generates an $\mathfrak{sl}_2$-subalgebra, and the Verma module construction (cf.[80]).

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[73] A generalisation to $G = U(1)^n$ is straightforward, and will be omitted.

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[86] L. A. Coburn, R. D. Moyer, and I. M. Singer, Acta Mathematica **130**, 279 (1973).
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